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TOPICAL REVIEW

Lagrangian submanifolds and dynamics on Lie algebroids

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Abstract

In some previous papers, a geometric description of Lagrangian mechanics on Lie algebroids has been developed. In this topical review, we give a Hamiltonian description of mechanics on Lie algebroids. In addition, we introduce the notion of a Lagrangian submanifold of a symplectic Lie algebroid and we prove that the Lagrangian (Hamiltonian) dynamics on Lie algebroids may be described in terms of Lagrangian submanifolds of symplectic Lie algebroids. The Lagrangian (Hamiltonian) formalism on Lie algebroids permits us to deal with Lagrangian (Hamiltonian) functions not defined necessarily on tangent (cotangent) bundles. Thus, we may apply our results to the projection of Lagrangian (Hamiltonian) functions which are invariant under the action of a symmetry Lie group. As a consequence, we obtain that Lagrange–Poincaré (Hamilton–Poincaré) equations are the Euler–Lagrange (Hamilton) equations associated with the corresponding Atiyah algebroid. Moreover, we prove that Lagrange–Poincaré (Hamilton–Poincaré) equations are the local equations defining certain Lagrangian submanifolds of symplectic Atiyah algebroids.

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Contents

1. Introduction

Lie algebroids have deserved a lot of interest in recent years. Since a Lie algebroid is a concept which unifies tangent bundles and Lie algebras, one can suspect their relation with mechanics. In his paper [50] Weinstein (see also the paper by Libermann [23]) developed a generalized theory of Lagrangian mechanics on Lie algebroids and obtained the equations of motion, using the linear Poisson structure on the dual of the Lie algebroid and the Legendre transformation associated with the Lagrangian L, when L is regular. In that paper, he also asks the question whether it is possible to develop a formalism similar on Lie algebroids to Klein's formalism [19] in ordinary Lagrangian mechanics. This task was finally done by Martínez [29] (see also [3, 8, 30, 31, 40]). The main notion is that of prolongation of a Lie algebroid over a mapping, introduced by Higgins and Mackenzie [17].

One could ask about the interest in generalizing classical mechanics on tangent and cotangent bundles to Lie algebroids. However, it is not a mere academic exercise. Indeed, if we apply our procedure to Atiyah algebroids we recover in a very natural way the Lagrange–Poincaré and Hamilton–Poincaré equations. In this case, the Lagrangian and Hamiltonian functions are not defined on tangent and cotangent bundles, but on the quotients by the structure Lie group (see section 9 of this paper). This fact is a good motivation for our study.

On the other hand, it is well known that Lagrangian submanifolds play an important role in the geometrical description of several aspects related to classical and quantum mechanics.

Thus, in [45, 46] Tulczyjew proved that it is possible to interpret the ordinary Lagrangian and Hamiltonian dynamics as Lagrangian submanifolds of convenient special symplectic manifolds. To do that, he introduced canonical isomorphisms which commute the tangent and cotangent functors. This construction is the so-called Tulczyjew triple for classical mechanics. Lagrangian submanifolds also arise in the reduction of Hamiltonian systems with a symmetry Lie group. In fact, momentum functions for a symplectic action of a Lie group are closely related to Lagrangian embeddings (see [48]). Hamilton–Jacobi theory and Lagrangian submanifolds are also closely related. Indeed, a generalized local (global) complete integral of the Hamilton–Jacobi equation for a Hamiltonian system, on the cotangent bundle T^*M of a manifold M, may be interpreted as a family of Lagrangian submanifolds of T^*M satisfying certain conditions (see [24]). On the other hand, intersection theory of Lagrangian submanifolds can be used to obtain results on boundary value problems for Hamiltonian flows (see [48]). In another direction, it is well known that Lagrangian foliations play an important role in the geometric quantization procedure of a quantizable symplectic manifold (see [22, 43]). In addition, symplectic groupoids and Lagrangian submanifolds are closely related (see [9]). We remark that symplectic groupoids may be used to integrate Poisson brackets (see [11]) and that the integrability problem for Poisson brackets is relevant to various quantization schemes (see [4, 21, 49]).

The purpose of this review is to give a description of Hamiltonian and Lagrangian dynamics on Lie algebroids in terms of Lagrangian submanifolds of symplectic Lie algebroids. The paper is organized as follows. In section 2.1, we recall the notion of prolongation $\mathcal{L}^f E$ of a Lie algebroid $\tau : E \longrightarrow M$ over a mapping $f : M' \longrightarrow M$; when f is just the canonical projection τ, then $\mathcal{L}^\tau E$ will play the role of the double tangent bundle. We also consider action Lie algebroids, which permit us to induce a Lie algebroid structure on the pull-back of a Lie algebroid by a mapping. The notion of quotient Lie algebroids is also discussed, and in particular, we consider Atiyah algebroids. In section 2.2 we develop the Lagrangian formalism on the prolongation $\mathcal{L}^\tau E$ starting with a Lagrangian function $L : E \longrightarrow \mathbb{R}$. Indeed, one can construct the Poincaré–Cartan 1- and 2-sections (i.e. $\theta_L \in \Gamma((\mathcal{L}^\tau E)^*)$ and $\omega_L \in \Gamma(\Lambda^2(\mathcal{L}^\tau E)^*)$, respectively) using the geometry of $\mathcal{L}^\tau E$ provided by the Euler section Δ and the vertical endomorphism S. The dynamics is given by a SODE ξ of $\mathcal{L}^\tau E$ (that is, a section ξ of $\mathcal{L}^\tau E$ such that $S\xi = \Delta$) satisfying $i_\xi \omega_L = d^{\mathcal{L}^\tau E} E_L$, where E_L is the energy associated with L (throughout this review d^E denotes the differential of the Lie algebroid E). As in classical mechanics, L is regular if and only if ω_L is a symplectic section, and in this case $\xi = \xi_L$ is uniquely defined and a SODE. Its solutions (curves in E) satisfy the Euler–Lagrange equations for L.

Sections 3.1–3.4 are devoted to developing a Hamiltonian description of mechanics on Lie algebroids. Now, the role of the cotangent bundle of the configuration manifold is played by the prolongation $\mathcal{L}^{\tau^*} E$ of E along the projection $\tau^* : E^* \longrightarrow M$, which is the dual bundle of E. We can construct the canonical Liouville 1-section λ_E and the canonical symplectic 2-section Ω_E on $\mathcal{L}^{\tau^*} E$. Theorem 3.4 and corollary 3.6 are the Lie algebroid version of the classical results concerning the universality of the standard Liouville 1-form on cotangent bundles. Given a Hamiltonian function $H : E^* \longrightarrow \mathbb{R}$, the dynamics are obtained solving the equation $i_{\xi_H} \Omega_E = d^{\mathcal{L}^{\tau^*} E} H$ with the usual notation. The solutions of ξ_H (curves in E^*) are those of the Hamilton equations for H. The Legendre transformation $Leg_L : E \longrightarrow E^*$ associated with a Lagrangian L induces a Lie algebroid morphism $\mathcal{L}Leg_L : \mathcal{L}^\tau E \longrightarrow \mathcal{L}^{\tau^*} E$, which permits in the regular case to connect Lagrangian and Hamiltonian formalisms as in classical mechanics. In section 3.5 we develop the corresponding Hamilton–Jacobi theory; we prove that the function $S : M \longrightarrow \mathbb{R}$ satisfying the Hamilton–Jacobi equation $d^E(H \circ d^E S) = 0$ is just the action for L.

As is well known, there is a canonical involution $\sigma_{TM} : TTM \longrightarrow TTM$ defined by Kobayashi [20]. In section 4, we prove that for an arbitrary Lie algebroid $\tau : E \longrightarrow M$ there is a unique Lie algebroid isomorphism $\sigma_E : \mathcal{L}^\tau E \longrightarrow \rho^*(TE)$ such that $\sigma_E^2 = id$, where $\mathcal{L}^\tau E$ is the prolongation of E by τ, and $\rho^*(TE)$ is the pull-back of the tangent bundle prolongation $T\tau : TE \longrightarrow TM$ via the anchor mapping $\rho : E \longrightarrow TM$ (theorem 4.4). Note that, as manifolds, $\rho^*(TE) = \mathcal{L}^\tau E$ and that, in addition, $\rho^*(TE)$ carries a Lie algebroid structure over E since the existence of an action of the tangent Lie algebroid $T\tau : TE \longrightarrow TM$ on τ. When E is the standard Lie algebroid TM we recover the standard canonical involution.

Section 5 is devoted to extending Tulczyjew's construction. First we define a canonical vector bundle isomorphism $\flat_{E^*} : \mathcal{L}^{\tau^*} E \longrightarrow (\mathcal{L}^{\tau^*} E)^*$ which is given using the canonical symplectic section of $\mathcal{L}^{\tau^*} E$. Next, using the canonical involution σ_E one defines a canonical vector bundle isomorphism $A_E : \mathcal{L}^{\tau^*} E \longrightarrow (\mathcal{L}^\tau E)^*$. Both vector bundle isomorphims extend the so-called Tulczyjew triple for classical mechanics.

In section 6 we introduce the notion of a symplectic Lie algebroid. The definition is the obvious one: Ω is a symplectic section on a Lie algebroid $\tau : E \longrightarrow M$ if it induces a nondegenerate bilinear form on each fibre of E and, in addition, it is d^E-closed ($d^E\Omega = 0$). In this case, the prolongation $\mathcal{L}^\tau E$ is symplectic too. The latter result extends the well-known result which proves that the tangent bundle of a symplectic manifold is also symplectic.

In section 7 we consider Lagrangian Lie subalgebroids of symplectic Lie algebroids; the definition is of course pointwise. This definition permits us to consider, in section 8, the notion of a Lagrangian submanifold of a symplectic Lie algebroid: a submanifold $i : S \longrightarrow E$ is a Lagrangian submanifold of the symplectic Lie algebroid $\tau : E \longrightarrow M$ with anchor map $\rho : E \longrightarrow TM$ if the following conditions hold:

- $\dim(\rho(E_{\tau^S(x)}) + (T_x\tau^S)(T_xS))$ does not depend on x, for all $x \in S$;
- the Lie subalgebroid $\mathcal{L}^{\tau^S} E$ of the symplectic Lie algebroid $\mathcal{L}^\tau E$ is Lagrangian;

here $\tau^S = \tau \circ i : S \longrightarrow M$. The classical results about Lagrangian submanifolds in symplectic geometry are extended to the present context in a natural way. Also, we generalize the interpretation of Tulzcyjew; for instance, given a Hamiltonian $H : E^* \longrightarrow \mathbb{R}$ we prove that $S_H = \xi_H(E^*)$ is a Lagrangian submanifold of the symplectic extension $\mathcal{L}^{\tau^*} E$ and that there exists a bijective correspondence between the admissible curves in S_H and the solutions of the Hamilton equations for H. For a Lagrangian $L : E \longrightarrow \mathbb{R}$ we prove that $S_L = \left(A_E^{-1} \circ d^{\mathcal{L}^\tau E} L\right)(E)$ is a Lagrangian submanifold of $\mathcal{L}^{\tau^*} E$ and, furthermore, that there exists a bijective correspondence between the admissible curves in S_L and the solutions of the Euler–Lagrange equations for L. In addition, we deduce that for a hyperregular Lagrangian L, then $S_{\xi_L} = \xi_L(E)$ is a Lagrangian submanifold of the symplectic extension $\mathcal{L}^\tau E$, and, moreover, we have that $\mathcal{L}Leg_L\left(S_{\xi_L}\right) = S_L = S_H$, for $H = E_L \circ Leg_L^{-1}$.

Sections 9.1–9.4 are devoted to some applications. Consider a (left) principal bundle $\pi : Q \longrightarrow M$ with structural group G. The Lie algebroid $\tau_Q|G : TQ/G \longrightarrow M$ is called the Atiyah algebroid associated with $\pi : Q \longrightarrow M$. One can prove that the prolongation $\mathcal{L}^{\tau_Q|G}(TQ/G)$ is isomorphic to the Atiyah algebroid associated with the principal bundle $\pi_T : TQ \longrightarrow TQ/G$, and, moreover, the dual vector bundle of $\mathcal{L}^{\tau_Q|G}(TQ/G)$ is isomorphic to the quotient vector bundle of $\pi_{TQ} : T^*(TQ) \longrightarrow TQ$ by the canonical lift action of G on $T^*(TQ)$. Similar results are obtained for cotangent bundles. In section 9.2 (respectively, section 9.3), we prove that the solutions of the Hamilton–Poincaré equations for a G-invariant Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R}$ (resp. the Lagrange–Poincaré equations for a G-invariant Lagrangian $L : TQ \longrightarrow \mathbb{R}$) are just the solutions of the Hamilton equations (resp. the Euler–Lagrange equations) on T^*Q/G for the reduced Hamiltonian $h : T^*Q/G \longrightarrow \mathbb{R}$ (resp. on TQ/G for the reduced Lagrangian $l : TQ/G \longrightarrow \mathbb{R}$). Moreover, in sections 9.2

and 9.3 all these equations are reinterpreted as those defining the corresponding Lagrangian submanifolds. In addition, in section 9.4 we show how our formalism allows us to obtain Wong's equations in a direct way.

Finally, in section 10 we discuss some open problems and the future work.

Manifolds are real, paracompact and C^∞. Maps are C^∞. Sum over crossed repeated indices is understood.

2. Lie algebroids and Lagrangian mechanics

2.1. Some algebraic constructions in the category of Lie algebroids

Let E be a vector bundle of rank n over a manifold M of dimension m and $\tau : E \to M$ be the vector bundle projection. Denote by $\Gamma(E)$ the $C^\infty(M)$-module of sections of $\tau : E \to M$. A *Lie algebroid structure* $(\llbracket\cdot,\cdot\rrbracket, \rho)$ on E is a Lie bracket $\llbracket\cdot,\cdot\rrbracket$ on the space $\Gamma(E)$ and a bundle map $\rho : E \to TM$, called *the anchor map*, such that if we also denote by $\rho : \Gamma(E) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then

$$\llbracket X, fY\rrbracket = f\llbracket X, Y\rrbracket + \rho(X)(f)Y,$$

for $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$. The triple $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ is called *a Lie algebroid over M* (see [25]).

If $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ is a Lie algebroid over M, then the anchor map $\rho : \Gamma(E) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(E), \llbracket\cdot,\cdot\rrbracket)$ and $(\mathfrak{X}(M), [\cdot,\cdot])$.

Trivial examples of Lie algebroids are real Lie algebras of finite dimension and the tangent bundle TM of an arbitrary manifold M.

Let $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ be a Lie algebroid over M. We consider the generalized distribution $\mathcal{F}^E$ on M whose characteristic space at a point $x \in M$ is given by

$$\mathcal{F}^E(x) = \rho(E_x)$$

where E_x is the fibre of E over x. The distribution $\mathcal{F}^E$ is finitely generated and involutive. Thus, $\mathcal{F}^E$ defines a generalized foliation on M in the sense of Sussmann [44]. $\mathcal{F}^E$ is *the Lie algebroid foliation* on M associated with E.

If $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ is a Lie algebroid, one may define *the differential of E*, $d^E : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^{k+1} E^*)$, as follows,

$$\begin{aligned} d^E\mu(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i \rho(X_i)(\mu(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \mu(\llbracket X_i, X_j\rrbracket, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned} \tag{2.1}$$

for $\mu \in \Gamma(\wedge^k E^*)$ and $X_0, \ldots, X_k \in \Gamma(E)$. It follows that $(d^E)^2 = 0$. Moreover, if X is a section of E, one may introduce, in a natural way, *the Lie derivative with respect to X*, as the operator $\mathcal{L}^E_X : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^k E^*)$ given by

$$\mathcal{L}^E_X = i_X \circ d^E + d^E \circ i_X.$$

Note that if $E = TM$ and $X \in \Gamma(E) = \mathfrak{X}(M)$ then d^{TM} and $\mathcal{L}^{TM}_X$ are the usual differential and the Lie derivative with respect to X, respectively.

If we take local coordinates (x^i) on M and a local basis $\{e_\alpha\}$ of sections of E, then we have the corresponding local coordinates (x^i, y^α) on E, where $y^\alpha(a)$ is the αth coordinate of $a \in E$ in the given basis. Such coordinates determine local functions ρ^i_α, $C^\gamma_{\alpha\beta}$ on M which contain the local information of the Lie algebroid structure, and accordingly they are called the *structure functions of the Lie algebroid*. They are given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \text{and} \qquad [\![e_\alpha, e_\beta]\!] = C^\gamma_{\alpha\beta} e_\gamma.$$

These functions should satisfy the relations

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta} \tag{2.2}$$

and

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\nu_{\alpha\mu} C^\mu_{\beta\gamma} \right] = 0, \tag{2.3}$$

which are usually called *the structure equations*.

If $f \in C^\infty(M)$, we have that

$$d^E f = \frac{\partial f}{\partial x^i} \rho^i_\alpha e^\alpha, \tag{2.4}$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$. On the other hand, if $\theta \in \Gamma(E^*)$ and $\theta = \theta_\gamma e^\gamma$ it follows that

$$d^E \theta = \left(\frac{\partial \theta_\gamma}{\partial x^i} \rho^i_\beta - \frac{1}{2} \theta_\alpha C^\alpha_{\beta\gamma} \right) e^\beta \wedge e^\gamma. \tag{2.5}$$

In particular,

$$d^E x^i = \rho^i_\alpha e^\alpha, \qquad d^E e^\alpha = -\tfrac{1}{2} C^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma.$$

On the other hand, if $(E, [\![\cdot,\cdot]\!], \rho)$ and $(E', [\![\cdot,\cdot]\!]', \rho')$ are Lie algebroids over M and M', respectively, then a morphism of vector bundles (F, f) of E on E'

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\tau"'] & E' \arrow[d, "\tau'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

is a Lie algebroid morphism if

$$d^E((F,f)^*\phi') = (F,f)^*(d^{E'}\phi'), \qquad \text{for all} \quad \phi' \in \Gamma(\wedge^k (E')^*) \ \text{ and for all } \ k. \tag{2.6}$$

Note that $(F, f)^*\phi'$ is the section of the vector bundle $\wedge^k E^* \to M$ defined by

$$((F,f)^*\phi')_x(a_1, \ldots, a_k) = \phi'_{f(x)}(F(a_1), \ldots, F(a_k)),$$

for $x \in M$ and $a_1, \ldots, a_k \in E_x$. We remark that (2.6) holds if and only if

$$\begin{aligned} d^E(g' \circ f) &= (F,f)^*(d^{E'} g'), && \text{for} \quad g' \in C^\infty(M'), \\ d^E((F,f)^*\alpha') &= (F,f)^*(d^{E'}\alpha'), && \text{for} \quad \alpha' \in \Gamma((E')^*). \end{aligned} \tag{2.7}$$

If $M = M'$ and $f = id : M \to M$ then, it is easy to prove that the pair (F, id) is a Lie algebroid morphism if and only if

$$F[\![X, Y]\!] = [\![FX, FY]\!]', \qquad \rho'(FX) = \rho(X)$$

for $X, Y \in \Gamma(E)$.

Other equivalent definitions of a Lie algebroid morphism may be found in [17]. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over M and E^* be the dual bundle to E. Then, E^* admits a linear Poisson structure Λ_{E^*}, that is, Λ_{E^*} is a 2-vector on E^* such that

$$[\Lambda_{E^*}, \Lambda_{E^*}] = 0$$

and if y and y' are linear functions on E^*, we have that $\Lambda_{E^*}(\mathrm{d}y, \mathrm{d}y')$ is also a linear function on E^*. If (x^i) are local coordinates on M, $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and (x^i, y_α) are the corresponding coordinates on E^* then the local expression of Λ_{E^*} is

$$\Lambda_{E^*} = \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta} + \rho^i_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^i}, \tag{2.8}$$

where ρ^i_α and $C^\gamma_{\alpha\beta}$ are the structure functions of E with respect to the coordinates (x^i) and to the basis $\{e_\alpha\}$. The Poisson structure Λ_{E^*} induces a linear Poisson bracket of functions on E^* which we will denote by $\{\,,\}_{E^*}$. In fact, if $F, G \in C^\infty(E^*)$ then

$$\{F, G\}_{E^*} = \Lambda_{E^*}(d^{TE^*}F, d^{TE^*}G).$$

On the other hand, if f is a function on M then the associated basic function $f^v \in C^\infty(E^*)$ defined by f is given by

$$f^v = f \circ \tau^*.$$

In addition, if X is a section of E then the linear function $\hat{X} \in C^\infty(E^*)$ defined by X is given by

$$\hat{X}(a^*) = a^*(X(\tau^*(a^*))), \qquad \text{for all} \quad a^* \in E^*.$$

The Poisson bracket $\{\,,\}_{E^*}$ is then characterized by the following relations,

$$\{f^v, g^v\}_{E^*} = 0, \qquad \{\hat{X}, g^v\}_{E^*} = (\rho(X)g)^v \qquad \text{and} \qquad \{\hat{X}, \hat{Y}\}_{E^*} = \widehat{[\![X, Y]\!]}, \tag{2.9}$$

for $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

In local coordinates (x^i, y_α) on E^* we have that

$$\{x^i, x^j\}_{E^*} = 0 \qquad \{y_\alpha, x^j\}_{E^*} = \rho^j_\alpha \qquad \text{and} \qquad \{y_\alpha, y_\beta\}_{E^*} = y_\gamma C^\gamma_{\alpha\beta},$$

(for more details, see [9, 10]).

2.1.1. The prolongation of a Lie algebroid over a smooth map. In this section, we will recall the definition of the Lie algebroid structure on the prolongation of a Lie algebroid over a smooth map. We will follow [17] (see section 1 in [17]).

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $f : M' \to M$ be a smooth map.

We consider the subset $\mathcal{L}^f E$ of $E \times TM'$ defined by

$$\mathcal{L}^f E = \{(b, v') \in E \times TM' / \rho(b) = (Tf)(v')\}$$

where $Tf : TM' \to TM$ is the tangent map to f.

Denote by $\tau^f : \mathcal{L}^f E \to M'$ the map given by

$$\tau^f(b, v') = \tau_{M'}(v'),$$

for $(b, v') \in \mathcal{L}^f E$, $\tau_{M'} : TM' \to M'$ being the canonical projection. If x' is a point of M', it follows that

$$(\tau^f)^{-1}(x') = (\mathcal{L}^f E)_{x'} = \{(b, v') \in E_{f(x')} \times T_{x'}M' / \rho(b) = (T_{x'}f)(v')\}$$

is a vector subspace of $E_{f(x')} \times T_{x'}M'$, where $E_{f(x')}$ is the fibre of E over the point $f(x') \in M$. Moreover, if m' is the dimension of M', one may prove that

$$\dim(\mathcal{L}^f E)_{x'} = n + m' - \dim(\rho(E_{f(x')}) + (T_{x'}f)(T_{x'}M')).$$

Thus, if we suppose that there exists $c \in \mathbb{N}$ such that

$$\dim(\rho(E_{f(x')}) + (T_{x'}f)(T_{x'}M')) = c, \qquad \text{for all} \quad x' \in M', \tag{2.10}$$

then we conclude that $\mathcal{L}^f E$ is a vector bundle over M' with vector bundle projection $\tau^f : \mathcal{L}^f E \to M'$.

Remark 2.1. If ρ and $T(f)$ are transversal, that is,

$$\rho(E_{f(x')}) + (T_{x'}f)(T_{x'}M') = T_{f(x')}M, \qquad \text{for all} \quad x' \in M', \tag{2.11}$$

then it is clear that (2.10) holds. Note that if E is a transitive Lie algebroid (that is, ρ is an epimorphism of vector bundles) or f is a submersion, we deduce that (2.11) also holds.

Next, we will assume that condition (2.10) holds and we will describe the sections of the vector bundle $\tau^f : \mathcal{L}^f E \to M'$.

Denote by f^*E the pullback of E over f, that is,

$$f^*E = \{(x', b) \in M' \times E / f(x') = \tau(b)\}.$$

f^*E is a vector bundle over M' with vector bundle projection

$$pr_1 : f^*E \to M', \qquad (x', b) \in f^*E \to x' \in M'.$$

Furthermore, if σ is a section of $pr_1 : f^*E \to M'$ then

$$\sigma = h'_i(X_i \circ f),$$

for suitable $h'_i \in C^\infty(M')$ and $X_i \in \Gamma(E)$.

On the other hand, if $X^\wedge$ is a section of the vector bundle $\tau^f : \mathcal{L}^f E \to M'$, one may prove that there exists a unique $\sigma \in \Gamma(f^*E)$ and a unique $X' \in \mathfrak{X}(M')$ such that

$$(T_{x'}f)(X'(x')) = \rho(\sigma(x')), \qquad \text{for all} \quad x' \in M', \tag{2.12}$$

and $X^\wedge(x') = (\sigma(x'), X'(x'))$. Thus,

$$X^\wedge(x') = (h'_i(x')X_i(f(x')), X'(x'))$$

for suitable $h'_i \in C^\infty(M')$, $X_i \in \Gamma(E)$ and, in addition,

$$(T_{x'}f)(X'(x')) = h'_i(x')\rho(X_i)(f(x')).$$

Conversely, if $\sigma \in \Gamma(f^*E)$ and $X' \in \mathfrak{X}(M')$ satisfy condition (2.12) then the map $X^\wedge : M' \to \mathcal{L}^f E$ given by

$$X^\wedge(x') = (\sigma(x'), X'(x')), \qquad \text{for all} \quad x' \in M',$$

is a section of the vector bundle $\tau^f : \mathcal{L}^f E \to M'$.

Now, we consider the homomorphism of $C^\infty(M')$-modules $\rho^f : \Gamma(\mathcal{L}^f E) \to \mathfrak{X}(M')$ and the Lie bracket $[\![\cdot,\cdot]\!]^f : \Gamma(\mathcal{L}^f E) \times \Gamma(\mathcal{L}^f E) \to \Gamma(\mathcal{L}^f E)$ on the space $\Gamma(\mathcal{L}^f E)$ defined as follows. If $X^\wedge \equiv (\sigma, X') \in \Gamma(f^*E) \times \mathfrak{X}(M')$ is a section of $\tau^f : \mathcal{L}^f E \to M'$ then

$$\rho^f(X^\wedge) = X' \tag{2.13}$$

and if $(h'_i(X_i \circ f), X')$ and $(s'_j(Y_j \circ f), Y')$ are two sections of $\tau^f : \mathcal{L}^f E \to M'$, with $h'_i, s'_j \in C^\infty(M')$, $X_i, Y_j \in \Gamma(E)$ and $X', Y' \in \mathfrak{X}(M')$, then

$$\begin{aligned}[\![(h'_i(X_i \circ f), X'), (s'_j(Y_j \circ f), Y')]\!]^f &= (h'_i s'_j([\![X_i, Y_j]\!] \circ f) \\ &\quad + X'(s'_j)(Y_j \circ f) - Y'(h'_i)(X_i \circ f), [X', Y']).\end{aligned} \tag{2.14}$$

The pair $([\![\cdot,\cdot]\!]^f, \rho^f)$ defines a Lie algebroid structure on the vector bundle $\tau^f : \mathcal{L}^f E \to M'$ (see [17]).

$(\mathcal{L}^f E, [\![\cdot,\cdot]\!]^f, \rho^f)$ is *the prolongation of the Lie algebroid E over the map f* (*the inverse-image Lie algebroid of E over f* in the terminology of [17]).

On the other hand, if $pr_1 : \mathcal{L}^f E \to E$ is the canonical projection on the first factor then the pair (pr_1, f) is a morphism between the Lie algebroids $(\mathcal{L}^f E, [\![\cdot,\cdot]\!]^f, \rho^f)$ and $(E, [\![\cdot,\cdot]\!], \rho)$ (for more details, see [17]).

2.1.2. Action Lie algebroids. In this section, we will recall the definition of the Lie algebroid structure of an action Lie algebroid. We will follow again [17].

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $f : M' \to M$ be a smooth map. Denote by f^*E the pull-back of E over f. f^*E is a vector bundle over M' whose vector bundle projection is the restriction to f^*E of the first canonical projection $pr_1 : M' \times E \to M'$.

However, f^*E is not, in general, a Lie algebroid over M'. Now, suppose that $\Psi : \Gamma(E) \to \mathfrak{X}(M')$ is an action of E on f, that is, Ψ is a $\mathbb{R}$-linear map which satisfies the following conditions:

(i) $\Psi(hX) = (h \circ f)\Psi X$,
(ii) $\Psi[\![X, Y]\!] = [\Psi X, \Psi Y]$,
(iii) $\Psi X(h \circ f) = \rho(X)(h) \circ f$,

for $X, Y \in \Gamma(E)$ and $h \in C^\infty(M)$. The action Ψ allows us to introduce a homomorphism of $C^\infty(M')$-modules $\rho_\Psi : \Gamma(f^*E) \to \mathfrak{X}(M')$ and a Lie bracket $[\![\cdot,\cdot]\!]_\Psi : \Gamma(f^*E) \times \Gamma(f^*E) \to \Gamma(f^*E)$ on the space $\Gamma(f^*E)$ defined as follows. If $\sigma = h'_i(X_i \circ f)$ and $\gamma = s'_j(Y_j \circ f)$ are sections of f^*E, with $h'_i, s'_j \in C^\infty(M')$ and $X_i, Y_j \in \Gamma(E)$, then

$$\rho_\Psi(\sigma) = h'_i\Psi(X_i),$$
$$[\![\sigma,\gamma]\!]_\Psi = h'_i s'_j([\![X_i, Y_j]\!] \circ f) + h'_i\Psi(X_i)(s'_j)(Y_j \circ f) - s'_j\Psi(Y_j)(h'_i)(X_i \circ f).$$

The pair $([\![\cdot,\cdot]\!]_\Psi, \rho_\Psi)$ defines a Lie algebroid structure on f^*E. The corresponding Lie algebroid is denoted by $E \ltimes M'$ or $E \ltimes f$ and we call it *an action Lie algebroid* (for more details, see [17]).

Remark 2.2. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M.

(i) *A Lie subalgebroid* is a morphism of Lie algebroids $j : F \to E, i : N \to M$ such that the pair (j, i) is a monomorphism of vector bundles and i is an injective inmersion (see [17]).

(ii) Suppose that $f : M' \to M$ is a smooth map and that $\Psi : \Gamma(E) \to \mathfrak{X}(M')$ is an action of E on f. The anchor map ρ_Ψ of f^*E induces a morphism between the vector bundles f^*E and TM' which we will also denote by ρ_Ψ. Thus, if $\mathcal{L}^f E$ is the prolongation of E over f, we may introduce the map

$$(id_E, \rho_\Psi) : f^*E \to \mathcal{L}^f E,$$

given by

$$(id_E, \rho_\Psi)(x', a) = (a, \rho_\Psi(x', a)),$$

for $(x', a) \in (f^*E)_{x'} \subseteq \{x'\} \times E_{f(x')}$, with $x' \in M'$. Moreover, if $id_{M'} : M' \to M'$ is the identity map then it is easy to prove that the pair $((id_E, \rho_\Psi), id_{M'})$ is a Lie subalgebroid. In fact, the map $(id_E, \rho_\Psi) : f^*E \to \mathcal{L}^f E$ is a section of the canonical projection

$$(id_E, \tau_{M'}) : \mathcal{L}^f E \to f^*E$$

defined by

$$(id_E, \tau_{M'})(a, X_{x'}) = (x', a)$$

for $(a, X_{x'}) \in (\mathcal{L}^f E)_{x'} \subseteq E_{f(x')} \times T_{x'}M'$, with $x' \in M'$.

2.1.3. Quotient Lie algebroids by the action of a Lie group. Let $\pi : Q \to M$ be a principal bundle with structural group G. Denote by $\phi : G \times Q \to Q$ the free action of G on Q.

Now, suppose that $\tilde{E}$ is a vector bundle over Q of rank n, with vector bundle projection $\tilde{\tau} : \tilde{E} \to Q$ and that $\tilde{\phi} : G \times \tilde{E} \to \tilde{E}$ is an action of G on $\tilde{E}$ such that:

(i) For each $g \in G$, the pair $(\tilde{\phi}_g, \phi_g)$ induces an isomorphism of vector bundles. Thus, the following diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{\phi}_g"] \arrow[d, "\tilde{\tau}"'] & \tilde{E} \arrow[d, "\tilde{\tau}"] \\ Q \arrow[r, "\phi_g"] & Q \end{tikzcd}$$

is commutative and for each $q \in Q$, the map

$$\tilde{\phi}_g : \tilde{E}_q \to \tilde{E}_{\phi_g(q)}$$

is a linear isomorphism between the vector spaces $\tilde{E}_q$ and $\tilde{E}_{\phi_g(q)}$.

(ii) $\tilde{E}$ is covered by the ranges of equivariant charts, that is, around each $q_0 \in Q$ there is a π-satured open set $\tilde{U} = \pi^{-1}(U)$, where $U \subseteq M$ is an open subset with $x_0 = \pi(q_0) \in U$ and a vector bundle chart $\tilde{\varphi} : \tilde{U} \times \mathbb{R}^n \to \tilde{\tau}^{-1}(\tilde{U})$ for $\tilde{E}$ which is equivariant in the sense that

$$\tilde{\varphi}(\phi_g(q), p) = \tilde{\phi}_g(\tilde{\varphi}(q, p))$$

for all $g \in G, q \in \tilde{U}$ and $p \in \mathbb{R}^n$.

Under conditions (i) and (ii), the orbit set $E = \tilde{E}/G$ has a unique vector bundle structure over $M = Q/G$ of rank n such that the pair $(\tilde{\pi}, \pi)$ is a morphism of vector bundles and $\tilde{\pi} : \tilde{E} \to E = \tilde{E}/G$ is a surjective submersion, where $\tilde{\pi} : \tilde{E} \to E = \tilde{E}/G$ is the canonical projection. The vector bundle projection $\tau = \tilde{\tau}|G : E \to M$ of E is given by

$$\tau[\tilde{u}] = [\tilde{\tau}(\tilde{u})], \qquad \text{for} \quad \tilde{u} \in \tilde{E}.$$

Moreover, if $q \in Q$ and $\pi(q) = x$ then the map

$$\tilde{\pi}_{|\tilde{E}_q} : \tilde{E}_q \to E_x, \qquad \tilde{u} \to [\tilde{u}]$$

is a linear isomorphism between the vector spaces $\tilde{E}_q$ and E_x.

We call (E, τ, M) *the quotient vector bundle of* $(\tilde{E}, \tilde{\tau}, Q)$ *by the action of* G (see [25]). On the other hand, a section $\tilde{X} : Q \to \tilde{E}$ of $\tilde{\tau} : \tilde{E} \to Q$ is said to be *invariant* if the map $\tilde{X}$ is equivariant, that is, the following diagram

$$\begin{tikzcd} Q \arrow[r, "\tilde{X}"] \arrow[d, "\phi_g"'] & \tilde{E} \arrow[d, "\tilde{\phi}_g"] \\ Q \arrow[r, "\tilde{X}"] & \tilde{E} \end{tikzcd}$$

is commutative, for all $g \in G$.

We will denote by $\Gamma(\tilde{E})^G$ the set of invariant sections of the vector bundle $\tilde{\tau} : \tilde{E} \to Q$. $\Gamma(\tilde{E})^G$ is a $C^\infty(M)$-module where

$$f\tilde{X} = (f \circ \pi)\tilde{X}, \qquad \text{for} \quad f \in C^\infty(M) \quad \text{and} \quad \tilde{X} \in \Gamma(\tilde{E})^G.$$

Furthermore, there exists an isomorphism between the $C^\infty(M)$-modules $\Gamma(E)$ and $\Gamma(\tilde{E})^G$. In fact, if $\tilde{X} \in \Gamma(\tilde{E})^G$ then the corresponding section $X \in \Gamma(E)$ is given by

$$X(x) = \tilde{\pi}(\tilde{X}(q)), \qquad \text{for} \quad x \in M,$$

with $q \in Q$ and $\pi(q) = x$ (for more details, see [25]).

Examples 2.3. (a) Suppose that $\tilde{E} = TQ$ and that $\tilde{\phi} : G \times TQ \to TQ$ is the tangent lift ϕ^T of ϕ defined by

$$\phi^T_g = T\phi_g, \qquad \text{for all} \quad g \in G.$$

Then, ϕ^T satisfies conditions (i) and (ii) and, thus, one may consider the quotient vector bundle $(E = TQ/G, \tau_Q|G, M)$ of (TQ, τ_Q, M) by the action of G. The space $\Gamma(TQ/G)$ may be identified with the set of vector fields on Q which are G-invariant.

(b) Assume that $\tilde{E} = T^*Q$ and that $\tilde{\phi} : G \times T^*Q \to T^*Q$ is the cotangent lift ϕ^{T^*} of ϕ defined by

$$\phi^{T^*}_g = T^*\phi_{g^{-1}}, \qquad \text{for all} \quad g \in G.$$

Then, ϕ^{T^*} satisfies conditions (i) and (ii) and, therefore, one may consider the quotient vector bundle $(T^*Q/G, \pi_Q|G, M)$ of (T^*Q, π_Q, Q) by the action of G. Moreover, if $(\tau_Q|G)^* : (TQ/G)^* \to M$ is the dual vector bundle to $\tau_Q|G : TQ/G \to M$ it is easy to prove that the vector bundles $(\tau_Q|G)^* : (TQ/G)^* \to M$ and $\pi_Q|G : T^*Q/G \to M$ are isomorphic.

(c) Suppose that $\mathfrak{g}$ is the Lie algebra of G, that $\tilde{E}$ is the trivial vector bundle $pr_1 : Q \times \mathfrak{g} \to Q$ and that the action $\tilde{\phi} = (\phi, Ad)$ of G on $Q \times \mathfrak{g}$ is given by

$$(\phi, Ad)_g(q, \xi) = (\phi_g(q), Ad_g\xi), \qquad \text{for} \quad g \in G \quad \text{and} \quad (q, \xi) \in Q \times \mathfrak{g}, \tag{2.15}$$

where $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of G on $\mathfrak{g}$. Note that the space $\Gamma(Q \times \mathfrak{g})$ may be identified with the set of π-vertical vector fields on Q. In addition, $\tilde{\phi}$ satisfies conditions (i) and (ii) and the resultant quotient vector bundle $pr_1|G : \tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G \to M = Q/G$ is just *the adjoint bundle associated with the principal bundle* $\pi : Q \to M$. Furthermore, if for each $\xi \in \mathfrak{g}$, we denote by ξ_Q the infinitesimal generator of the action ϕ associated with ξ, then the map

$$j : \tilde{\mathfrak{g}} \to TQ/G, \qquad [(q, \xi)] \to [\xi_Q(q)]$$

induces a monomorphism between the vector bundles $\tilde{\mathfrak{g}}$ and TQ/G. Thus, $\tilde{\mathfrak{g}}$ may be considered as a vector subbundle of TQ/G. In addition, the space $\Gamma(\tilde{\mathfrak{g}})$ may be identified with the set of vector fields on Q which are vertical and G-invariant (see [25]).

Remark 2.4. (a) The tangent map to π, $T\pi : TQ \to TM$, induces an epimorphism $[T\pi] : TQ/G \to TM$, between the vector bundles TQ/G and TM and, furthermore, $Imj = \ker[T\pi]$. Therefore, we have an exact sequence of vector bundles

$$\tilde{\mathfrak{g}} \xrightarrow{\ j\ } TQ/G \xrightarrow{\ [T\pi]\ } TM$$

which is just *the Atiyah sequence associated with the principal bundle* $\pi : Q \to M$ (for more details, see [25]).

(b) Recall that if $\pi : Q \to M$ is a principal bundle with structural group G then a principal connection A on Q is a Lie algebra-valued one-form $A : TQ \to \mathfrak{g}$ such that:

(i) For all $\xi \in \mathfrak{g}$ and for all $q \in Q$, $A(\xi_Q(q)) = \xi$, and
(ii) A is equivariant with respect to the actions $\phi^T : G \times TQ \to TQ$ and $Ad : G \times \mathfrak{g} \to \mathfrak{g}$.

Any choice of a connection in the principal bundle $\pi : Q \to M$ determines an isomorphism between the vector bundles $TQ/G \to M$ and $TM \oplus \tilde{\mathfrak{g}} \to M$. In fact, if $A : TQ \to \mathfrak{g}$ is a principal connection then the map $I_A : TQ/G \to TM \oplus \tilde{\mathfrak{g}}$ defined by

$$I_A[\tilde{X}_q] = (T_q\pi)(\tilde{X}_q) \oplus [(q, A(\tilde{X}_q))] \tag{2.16}$$

for $\tilde{X}_q \in T_qQ$, is a vector bundle isomorphism over the identity $id : M \to M$ (see [6, 25]).

Next, using the principal connection A, we will obtain a local basis of $\Gamma(TQ/G) \cong \Gamma(TM \oplus \tilde{\mathfrak{g}}) \cong \mathfrak{X}(M) \oplus \Gamma(\tilde{\mathfrak{g}})$. First of all, we choose a local trivialization of the principal bundle $\pi : Q \to M$ to be $U \times G$, where U is an open subset of M. Thus, we consider the trivial principal bundle $\pi : U \times G \to U$ with structural group G acting only on the second factor by left multiplication. Let e be the identity element of G and assume that there are local coordinates (x^i) in U and that $\{\xi_a\}$ is a basis of $\mathfrak{g}$. Denote by $\{\xi_a^L\}$ the corresponding left-invariant vector fields on G, that is,

$$\xi_a^L(g) = (T_eL_g)(\xi_a), \qquad \text{for} \quad g \in G,$$

where $L_g : G \to G$ is the left translation by g, and suppose that

$$A\left(\frac{\partial}{\partial x^i}_{|(x,e)}\right) = A_i^a(x)\xi_a,$$

for $i \in \{1, \dots, m\}$ and $x \in U$. Then, the horizontal lift of the vector field $\frac{\partial}{\partial x^i}$ on U is the vector field $\left(\frac{\partial}{\partial x^i}\right)^h$ on $U \times G$ given by

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - A_i^a\xi_a^L.$$

Therefore, the vector fields on $U \times G$

$$\left\{e_i = \frac{\partial}{\partial x^i} - A_i^a\xi_a^L, e_b = \xi_b^L\right\} \tag{2.17}$$

are G-invariant and they define a local basis $\{e_i', e_b'\}$ of $\Gamma(TQ/G) \cong \mathfrak{X}(M) \oplus \Gamma(\tilde{\mathfrak{g}})$. We will denote by (x^i, y^i, y^b) the corresponding fibred coordinates on TQ/G. In the terminology of [6],

$$y^i = \dot{x}^i, \qquad y^b = \bar{v}^b, \qquad \text{for} \quad i \in \{1, \dots, m\} \text{ and } b \in \{1, \dots, n\}.$$

Now, we will return to the general case.

Assume that $\pi : Q \to M$ is a principal bundle with structural group G, that $\tilde{E}$ is a vector bundle over Q of rank n with vector bundle projection $\tilde{\tau} : \tilde{E} \to Q$ and that $\tilde{\phi} : G \times \tilde{E} \to \tilde{E}$ is an action of G on $\tilde{E}$ which satisfies conditions (i) and (ii). Denote by $\phi : G \times Q \to Q$ the free action of G on Q.

We will also suppose that $(\tilde{[\![\cdot, \cdot]\!]}, \tilde{\rho})$ is a Lie algebroid structure on $\tilde{\tau} : \tilde{E} \to Q$ such that the space $\Gamma(\tilde{E})^G$ is a Lie subalgebra of the Lie algebra $(\Gamma(\tilde{E}), \tilde{[\![\cdot, \cdot]\!]})$. Thus, one may define a Lie algebra structure

$$[\![\cdot, \cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

on $\Gamma(E)$. The Lie bracket $[\![\cdot, \cdot]\!]$ is the restriction of $\tilde{[\![\cdot, \cdot]\!]}$ to $\Gamma(\tilde{E})^G \cong \Gamma(E)$.

On the other hand, the anchor map $\tilde{\rho} : \tilde{E} \to TQ$ is equivariant. In fact, if $\tilde{X} \in \Gamma(\tilde{E})^G$, $f \in C^\infty(M)$ and $\tilde{Y} \in \Gamma(\tilde{E})^G$ then

$$\tilde{[\![\tilde{X}, (f \circ \pi)\tilde{Y}]\!]} = (f \circ \pi)\tilde{[\![\tilde{X}, \tilde{Y}]\!]} + \tilde{\rho}(\tilde{X})(f \circ \pi)\tilde{Y}$$

is an invariant section. This implies that the function $\tilde{\rho}(\tilde{X})(f \circ \pi)$ is projectable, that is, there exists $\rho(\tilde{X})(f) \in C^\infty(M)$ such that

$$\tilde{\rho}(\tilde{X})(f \circ \pi) = \rho(\tilde{X})(f) \circ \pi, \qquad \forall f \in C^\infty(M).$$

The map $\rho(\tilde{X}) : C^\infty(M) \to C^\infty(M)$ defines a vector field $\rho(\tilde{X})$ on M and $\tilde{\rho}(\tilde{X})$ is π-projectable onto $\rho(\tilde{X})$.

This proves that $\tilde{\rho} : \tilde{E} \to TQ$ is equivariant and, therefore, $\tilde{\rho}$ induces a bundle map $\rho : E = \tilde{E}/G \to TM = T(Q/G)$ such that the following diagram is commutative

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{\rho}"] \arrow[d, "\tilde{\pi}"'] & TQ \arrow[d, "T\pi"] \\ E \arrow[r, "\rho"] & TM \end{tikzcd}$$

Moreover, it follows that the pair $([\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid structure on the quotient vector bundle $\tau = \tilde{\tau}|G : E = \tilde{E}/G \to M = Q/G$. In addition, from the definition of $([\![\cdot,\cdot]\!], \rho)$, one deduces that the pair $(\tilde{\pi}, \pi)$ is a morphism between the Lie algebroids $(\tilde{E}, \tilde{[\![\cdot,\cdot]\!]}, \tilde{\rho})$ and $(E, [\![\cdot,\cdot]\!], \rho)$.

We call $(E, [\![\cdot,\cdot]\!], \rho)$ *the quotient Lie algebroid of* $(\tilde{E}, \tilde{[\![\cdot,\cdot]\!]}, \tilde{\rho})$ *by the action of the Lie group* G (a more general definition of a quotient Lie algebroid may be found in [17]).

Examples 2.5. (i) Assume that $\tilde{E} = TQ$ and that $\tilde{\phi}$ is the tangent action $\phi^T : G \times TQ \to TQ$. Consider on the vector bundle $\tau_Q : TQ \to Q$ the standard Lie algebroid structure $([\cdot,\cdot], id)$. Since the Lie bracket of two G-invariant vector fields on Q is also G-invariant, we obtain a Lie algebroid structure $([\![\cdot,\cdot]\!], \rho)$ on the quotient vector bundle $\tau_Q|G : E = TQ/G \to M = Q/G$. We call $(E = TQ/G, [\![\cdot,\cdot]\!], \rho)$ the *Atiyah algebroid associated with the principal bundle* $\pi : Q \to M$ (see [25]).

(ii) Suppose that $\tilde{E}$ is the trivial vector bundle $pr_1 : Q \times \mathfrak{g} \to Q$. The space $\Gamma(Q \times \mathfrak{g})$ is isomorphic to the space of π-vertical vector fields on Q and, thus, $\Gamma(Q \times \mathfrak{g})$ is a Lie subalgebra of $(\mathfrak{X}(Q), [\cdot,\cdot])$. This implies that the vector bundle $pr_1 : Q \times \mathfrak{g} \to \mathfrak{g}$ admits a Lie algebroid structure $(\tilde{[\![\cdot,\cdot]\!]}, \tilde{\rho})$. On the other hand, denote by (ϕ, Ad) the action of G on $Q \times \mathfrak{g}$ given by (2.15). Then, the space $\Gamma(Q \times \mathfrak{g})^G$ is isomorphic to the space $\mathfrak{X}^v(Q)^G$ of π-vertical G-invariant vector fields on Q. Since $\mathfrak{X}^v(Q)^G$ is a Lie subalgebra of $(\mathfrak{X}(Q), [\cdot,\cdot])$, one may define a Lie algebroid structure on the adjoint bundle $pr_1|G : \tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G \to M = Q/G$ with anchor map $\rho = 0$, that is, the adjoint bundle is a Lie algebra bundle (see [25]).

Now, let $A : TQ \to \mathfrak{g}$ be a connection in the principal bundle $\pi : Q \to M$ and $B : TQ \oplus TQ \to \mathfrak{g}$ be the curvature of A. Using the principal connection A one may identity the vector bundles $E = TQ/G \to M = Q/G$ and $TM \oplus \tilde{\mathfrak{g}} \to M$, via the isomorphism I_A given by (2.16). Under this identification, the Lie bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(TQ/G) \cong \Gamma(TM \oplus \tilde{\mathfrak{g}}) \cong \mathfrak{X}(M) \oplus \mathfrak{X}^v(Q)^G$ is given by

$$[\![X \oplus \tilde{\xi}, Y \oplus \tilde{\eta}]\!] = [X, Y] \oplus ([\tilde{\xi}, \tilde{\eta}] + [X^h, \tilde{\eta}] - [Y^h, \tilde{\xi}] - B(X^h, Y^h)),$$

for $X, Y \in \mathfrak{X}(M)$ and $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}^v(Q)^G$, where $X^h \in \mathfrak{X}(Q)$ (respectively, $Y^h \in \mathfrak{X}(Q)$) is the horizontal lift of X (respectively, Y), via the principal connection A (see [6]). The anchor map $\rho : \Gamma(TQ/G) \cong \mathfrak{X}(M) \oplus \mathfrak{X}^v(Q)^G \to \mathfrak{X}(M)$ is given by

$$\rho(X \oplus \tilde{\xi}) = X.$$

Next, using the connection A, we will obtain the (local) structure functions of $(E, [\![\cdot,\cdot]\!], \rho)$ with respect to a local trivialization of the vector bundle.

First of all, we choose a local trivialization $U \times G$ of the principal bundle $\pi : Q \to M$, where U is an open subset of M such that there are local coordinates (x^i) on U. We will also suppose that $\{\xi_a\}$ is a basis of $\mathfrak{g}$ and that

$$A\left(\frac{\partial}{\partial x^i}_{|(x,e)}\right) = A_i^a(x)\xi_a, \qquad B\left(\frac{\partial}{\partial x^i}_{|(x,e)}, \frac{\partial}{\partial x^j}_{|(x,e)}\right) = B_{ij}^a(x)\xi_a, \tag{2.18}$$

for $i, j \in \{1, \ldots, m\}$ and $x \in U$. If c_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_a\}$ then

$$B_{ij}^c = \frac{\partial A_i^c}{\partial x^j} - \frac{\partial A_j^c}{\partial x^i} - c_{ab}^c A_i^a A_j^b. \tag{2.19}$$

Moreover, if $\{e_i', e_b'\}$ is the local basis of $\Gamma(TQ/G)$ considered in remark 2.4 (see (2.17)) then, using (2.19), we deduce that

$$[\![e_i', e_j']\!] = -B_{ij}^c e_c', \qquad [\![e_i', e_a']\!] = c_{ab}^c A_i^b e_c', \qquad [\![e_a', e_b']\!] = c_{ab}^c e_c',$$
$$\rho(e_i') = \frac{\partial}{\partial x^i}, \qquad \rho(e_a') = 0,$$

for $i, j \in \{1, \ldots, m\}$ and $a, b \in \{1, \ldots, n\}$. Thus, the local structure functions of the Atiyah algebroid $\tau_Q|G : E = TQ/G \to M = Q/G$ with respect to the local coordinates (x^i) and to the local basis $\{e_i', e_a'\}$ of $\Gamma(TQ/G)$ are

$$\begin{aligned} &C_{ij}^k = C_{ia}^j = -C_{ai}^j = C_{ab}^i = 0, \qquad C_{ij}^a = -B_{ij}^a, \qquad C_{ia}^c = -C_{ai}^c = c_{ab}^c A_i^b, \\ &C_{ab}^c = c_{ab}^c, \qquad \rho_i^j = \delta_{ij}, \qquad \rho_i^a = \rho_a^i = \rho_a^b = 0. \end{aligned} \tag{2.20}$$

On the other hand, as we know the dual vector bundle to the Atiyah algebroid is the quotient vector bundle $\pi_Q|G : T^*Q/G \to M = Q/G$ of the cotangent bundle $\pi_Q : T^*Q \to Q$ by the cotangent action ϕ^{T^*} of G on T^*Q (see example 2.3). Now, let Ω_{TQ} be the canonical symplectic 2-form of T^*Q and Λ_{TQ} be the Poisson 2-vector on T^*Q associated with Ω_Q. If (q^α) are local coordinates on Q and (q^α, p_α) are the corresponding fibred coordinates on T^*Q then

$$\Omega_{TQ} = \mathrm{d}q^\alpha \wedge \mathrm{d}p_\alpha, \qquad \Lambda_{TQ} = \frac{\partial}{\partial p_\alpha} \wedge \frac{\partial}{\partial q^\alpha}.$$

Note that Λ_{TQ} is the linear Poisson structure on T^*Q associated with the standard Lie algebroid $\tau_Q : TQ \to Q$. In addition, it is well known that the cotangent action ϕ^{T^*} is symplectic, that is,

$$(\phi^{T^*})_g : (T^*Q, \Omega_{TQ}) \to (T^*Q, \Omega_{TQ})$$

is a symplectomorphism, for all $g \in G$. Thus, the 2-vector Λ_{TQ} on TQ is G-invariant and it induces a 2-vector $\widetilde{\Lambda_{TQ}}$ on the quotient manifold T^*Q/G. Under the identification between the vector bundles $(\tau_Q|G)^* : E^* = (TQ/G)^* \to M = Q/G$ and $\pi_Q|G : T^*Q/G \to M = Q/G$, $\widetilde{\Lambda_{TQ}}$ is just the linear Poisson structure $\Lambda_E = \Lambda_{TQ/G}$ on $E^* = (TQ/G)^*$ associated with the Atiyah algebroid $\tau_Q|G : TQ/G \to M = Q/G$. If $(x^i, \dot{x}^i, \bar{v}^a)$ are the local coordinates on the vector bundle TQ/G considered in remark 2.4 and $(x^i, p_i, \bar{p}_a)$ are the corresponding coordinates on the dual vector bundle $(TQ/G)^* \cong T^*Q/G$ then, using (2.8) and (2.20), we obtain that the local expression of $\Lambda_{TQ/G}$ is

$$\Lambda_{TQ/G} = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i} + c_{ab}^c A_i^b \bar{p}_c \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial \bar{p}_a} + \frac{1}{2}\left(c_{ab}^c \bar{p}_c \frac{\partial}{\partial \bar{p}_a} \wedge \frac{\partial}{\partial \bar{p}_b} - B_{ij}^c \bar{p}_c \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}\right).$$

2.2. Lagrangian mechanics on Lie algebroids

In this section, we will recall some results about a geometric description of Lagrangian mechanics on Lie algebroids which has been developed by Martínez in [29].

2.2.1. The prolongation of a Lie algebroid over the vector bundle projection. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $\tau : E \to M$ be the vector bundle projection.

If $f \in C^\infty(M)$ we will denote by f^c and f^v *the complete and vertical lift* to E of f. f^c and f^v are the real functions on E defined by

$$f^c(a) = \rho(a)(f), \qquad f^v(a) = f(\tau(a)), \tag{2.21}$$

for all $a \in E$.

Now, let X be a section of E. Then, we can consider *the vertical lift of* X as the vector field on E given by

$$X^v(a) = X(\tau(a))^v_a, \qquad \text{for} \quad a \in E,$$

where ${}^v_a : E_{\tau(a)} \to T_a(E_{\tau(a)})$ is the canonical isomorphism between the vector spaces $E_{\tau(a)}$ and $T_a(E_{\tau(a)})$.

On the other hand, there exists a unique vector field X^c on E, *the complete lift of* X, satisfying the two following conditions:

(i) X^c is τ-projectable on $\rho(X)$ and
(ii) $X^c(\hat{\alpha}) = \widehat{\mathcal{L}^E_X \alpha}$,

for all $\alpha \in \Gamma(E^*)$ (see [13, 14]). Here, if $\beta \in \Gamma(E^*)$ then $\hat{\beta}$ is the linear function on E defined by

$$\hat{\beta}(b) = \beta(\tau(b))(b), \qquad \text{for all} \quad b \in E.$$

We have that (see [13, 14])

$$[X^c, Y^c] = [\![X, Y]\!]^c, \qquad [X^c, Y^v] = [\![X, Y]\!]^v, \qquad [X^v, Y^v] = 0. \tag{2.22}$$

Next, we consider the prolongation $\mathcal{L}^\tau E$ of E over the projection τ (see section 2.1.1). $\mathcal{L}^\tau E$ is a vector bundle over E of rank $2n$. Moreover, we may introduce *the vertical lift* $X^{\mathbf{v}}$ and *the complete lift* $X^{\mathbf{c}}$ of a section $X \in \Gamma(E)$ as the sections of $\mathcal{L}^\tau E \to E$ given by

$$X^{\mathbf{v}}(a) = (0, X^v(a)), \qquad X^{\mathbf{c}}(a) = (X(\tau(a)), X^c(a)) \tag{2.23}$$

for all $a \in E$. If $\{X_i\}$ is a local basis of $\Gamma(E)$ then $\{X_i^{\mathbf{v}}, X_i^{\mathbf{c}}\}$ is a local basis of $\Gamma(\mathcal{L}^\tau E)$ (see [29]).

Now, denote by $([\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ the Lie algebroid structure on $\mathcal{L}^\tau E$ (see section 2.1.1). It follows that

$$\begin{aligned}
&[\![X^{\mathbf{c}}, Y^{\mathbf{c}}]\!]^\tau = [\![X, Y]\!]^{\mathbf{c}}, && [\![X^{\mathbf{c}}, Y^{\mathbf{v}}]\!]^\tau = [\![X, Y]\!]^{\mathbf{v}}, && [\![X^{\mathbf{v}}, Y^{\mathbf{v}}]\!]^\tau = 0,\\
&\rho^\tau(X^{\mathbf{c}})(f^c) = (\rho(X)(f))^c, && \rho^\tau(X^{\mathbf{c}})(f^v) = (\rho(X)(f))^v, &&\\
&\rho^\tau(X^{\mathbf{v}})(f^c) = (\rho(X)(f))^v, && \rho^\tau(X^{\mathbf{v}})(f^v) = 0, &&
\end{aligned} \tag{2.24}$$

for $X, Y \in \Gamma(E)$ (see [29]).

Two other canonical objects on $\mathcal{L}^\tau E$ are *the Euler section* Δ and *the vertical endomorphism* S. Δ is the section of $\mathcal{L}^\tau E \to E$ defined by

$$\Delta(a) = (0, a^v_a), \qquad \text{for all} \quad a \in E$$

and S is the section of the vector bundle $(\mathcal{L}^\tau E) \otimes (\mathcal{L}^\tau E)^* \to E$ characterized by the following conditions

$$S(X^{\mathbf{v}}) = 0, \qquad S(X^{\mathbf{c}}) = X^{\mathbf{v}},$$

for $X \in \Gamma(E)$.

Finally, a section ξ of $\mathcal{L}^\tau E \to E$ is said to be a *second-order differential equation* (SODE) on E if $S(\xi) = \Delta$ (for more details, see [29]).

Remark 2.6. If E is the standard Lie algebroid TM, then $\mathcal{L}^\tau E = T(TM)$ and the Lie algebroid structure $([\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ is the usual one on the vector bundle $T(TM) \to TM$. Moreover, Δ is the Euler vector field on TM and S is the vertical endomorphism on TM.

Remark 2.7. Suppose that (x^i) are coordinates on an open subset U of M and that $\{e_\alpha\}$ is a basis of sections of $\tau^{-1}(U) \to U$. Denote by (x^i, y^α) the corresponding coordinates on $\tau^{-1}(U)$ and by ρ^i_α and $C^\gamma_{\alpha\beta}$ the corresponding structure functions of E. If X is a section of E and on U

$$X = X^\alpha e_\alpha$$

then X^v and X^c are the vector fields on E given by

$$X^v = X^\alpha \frac{\partial}{\partial y^\alpha}, \qquad X^c = X^\alpha \rho^i_\alpha \frac{\partial}{\partial x^i} + \left(\rho^i_\beta \frac{\partial X^\alpha}{\partial x^i} - X^\gamma C^\alpha_{\gamma\beta}\right) y^\beta \frac{\partial}{\partial y^\alpha}. \tag{2.25}$$

In particular,

$$e^v_\alpha = \frac{\partial}{\partial y^\alpha}, \qquad e^c_\alpha = \rho^i_\alpha \frac{\partial}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial}{\partial y^\gamma}. \tag{2.26}$$

On the other hand, if V_α, T_α are the sections of $\mathcal{L}^\tau E$ defined by $V_\alpha = e^{\mathbf{v}}_\alpha$, $T_\alpha = e^{\mathbf{c}}_\alpha$ then

$$X^{\mathbf{v}} = X^\alpha V_\alpha, \qquad X^{\mathbf{c}} = \left(\rho^i_\beta \frac{\partial X^\alpha}{\partial x^i} y^\beta\right) V_\alpha + X^\alpha T_\alpha. \tag{2.27}$$

Thus,

$$\begin{aligned} &[\![T_\alpha, T_\beta]\!]^\tau = \left(\rho^i_\delta \frac{\partial C^\gamma_{\alpha\beta}}{\partial x^i} y^\delta\right) V_\gamma + C^\gamma_{\alpha\beta} T_\gamma, \\ &[\![T_\alpha, V_\beta]\!]^\tau = C^\gamma_{\alpha\beta} V_\gamma, \qquad [\![V_\alpha, V_\beta]\!]^\tau = 0. \end{aligned} \tag{2.28}$$

The local expressions of Δ and S are the following ones,

$$\Delta = y^\alpha V_\alpha, \qquad S = T^\alpha \otimes V_\alpha, \tag{2.29}$$

where $\{T^\alpha, V^\alpha\}$ is the dual basis of $\{T_\alpha, V_\alpha\}$. Therefore, a section ξ of $\mathcal{L}^\tau E$ is a SODE if and only if the local expression of ξ is of the form

$$\xi = y^\alpha T_\alpha + \xi^\alpha V_\alpha,$$

where ξ^α are arbitrary local functions on E.

Note that

$$\begin{aligned} d^{\mathcal{L}^\tau E} f &= \left(\rho^i_\alpha \frac{\partial f}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial f}{\partial y^\gamma}\right) T^\alpha + \frac{\partial f}{\partial y^\alpha} V^\alpha, \\ d^{\mathcal{L}^\tau E} T^\gamma &= -\frac{1}{2} C^\gamma_{\alpha\beta} T^\alpha \wedge T^\beta, \\ d^{\mathcal{L}^\tau E} V^\gamma &= -\frac{1}{2}\left(\rho^i_\delta \frac{\partial C^\gamma_{\alpha\beta}}{\partial x^i} y^\delta\right) T^\alpha \wedge T^\beta + C^\gamma_{\alpha\beta} T^\alpha \wedge V^\beta, \end{aligned} \tag{2.30}$$

for $f \in C^\infty(E)$ and $\gamma \in \{1, \ldots, n\}$.

We also remark that there exists another local basis of sections on $\mathcal{L}^\tau E$. In fact, we may define the local section $\tilde{T}_\alpha$ as follows:

$$\tilde{T}_\alpha(a) = \left(e_\alpha(\tau(a)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|a}\right), \qquad \text{for all} \quad a \in \tau^{-1}(U). \tag{2.31}$$

Then, $\{\tilde{T}_\alpha, \tilde{V}_\alpha = V_\alpha\}$ is a local basis of sections of $\mathcal{L}^\tau E$.

Note that

$$\tilde{T}_\alpha = T_\alpha + C^\gamma_{\alpha\beta} y^\beta V_\gamma \tag{2.32}$$

and thus,

$$\begin{aligned} &[\![\tilde{T}_\alpha, \tilde{T}_\beta]\!]^\tau = C^\gamma_{\alpha\beta}\tilde{T}_\gamma, \qquad &&[\![\tilde{T}_\alpha, \tilde{V}_\beta]\!]^\tau = 0, \qquad &&[\![\tilde{V}_\alpha, \tilde{V}_\beta]\!]^\tau = 0, \\ &\rho^\tau(\tilde{T}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, &&\rho^\tau(\tilde{V}_\alpha) = \frac{\partial}{\partial y^\alpha}, \end{aligned} \tag{2.33}$$

for all α and β. Using the local basis $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ one may introduce, in a natural way, local coordinates $(x^i, y^\alpha; z^\alpha, v^\alpha)$ on $\mathcal{L}^\tau E$. If ω is a point of $(\tau^\tau)^{-1}(\tau^{-1}(U))$ ($\tau^\tau : \mathcal{L}^\tau E \to E$ being the vector bundle projection) then (x^i, y^α) are the coordinates of the point $\tau^\tau(\omega) \in \tau^{-1}(U)$ and

$$\omega = z^\alpha \tilde{T}_\alpha(\tau^\tau(\omega)) + v^\alpha \tilde{V}_\alpha(\tau^\tau(\omega)).$$

In addition, the anchor map ρ^τ is given by

$$\rho^\tau(x^i, y^\alpha; z^\alpha, v^\alpha) = \left(x^i, y^\alpha; \rho^i_\alpha z^\alpha, v^\alpha\right) \tag{2.34}$$

and if $\{\tilde{T}^\alpha, \tilde{V}^\alpha\}$ is the dual basis of $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ then

$$S = \tilde{T}^\alpha \otimes \tilde{V}_\alpha \tag{2.35}$$

and

$$\begin{aligned} &d^{\mathcal{L}^\tau E} f = \rho^i_\gamma \frac{\partial f}{\partial x^i}\tilde{T}^\gamma + \frac{\partial f}{\partial y^\gamma}\tilde{V}^\gamma, \\ &d^{\mathcal{L}^\tau E}\tilde{T}^\gamma = -\frac{1}{2}C^\gamma_{\alpha\beta}\tilde{T}^\alpha \wedge \tilde{T}^\beta, \qquad d^{\mathcal{L}^\tau E}\tilde{V}^\gamma = 0. \end{aligned} \tag{2.36}$$

2.2.2. The Lagrangian formalism on Lie algebroids. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $L : E \to \mathbb{R}$ be a Lagrangian function.

In this section, we will develop a geometric framework, which allows us to write the Euler–Lagrange equations associated with the Lagrangian function L in an intrinsic way (see [29]).

First of all, we introduce *the Poincaré–Cartan 1-section* $\theta_L \in \Gamma((\mathcal{L}^\tau E)^*)$ associated with L defined by

$$\theta_L(a)(\hat{X}_a) = \left(d^{\mathcal{L}^\tau E} L(a)\right)(S_a(\hat{X}_a)) = \rho^\tau(S_a(\hat{X}_a))(L) \tag{2.37}$$

for $a \in E$ and $\hat{X}_a \in (\mathcal{L}^\tau E)_a$, $(\mathcal{L}^\tau E)_a$ being the fibre of $\mathcal{L}^\tau E \to E$ over the point a. Then, *the Poincaré–Cartan 2-section* ω_L associated with L is, up to the sign, the differential of θ_L, that is,

$$\omega_L = -d^{\mathcal{L}^\tau E}\theta_L \tag{2.38}$$

and *the energy function* E_L is

$$E_L = \rho^\tau(\Delta)(L) - L. \tag{2.39}$$

Now, let $\gamma : I = (-\varepsilon, \varepsilon) \subseteq \mathbb{R} \to E$ be a curve in E. Then, γ is *a solution of the Euler–Lagrange equations* associated with L if and only if:

(i) γ is admissible, that is, $(\gamma(t), \dot{\gamma}(t)) \in (\mathcal{L}^\tau E)_{\gamma(t)}$, for all t.
(ii) $i_{(\gamma(t),\dot{\gamma}(t))}\omega_L(\gamma(t)) = \big(d^{\mathcal{L}^\tau E} E_L\big)(\gamma(t))$, for all t.

If (x^i) are coordinates on M, $\{e_\alpha\}$ is a local basis of $\Gamma(E)$, (x^i, y^α) are the corresponding coordinates on E and

$$\gamma(t) = (x^i(t), y^\alpha(t)),$$

then γ is a solution of the Euler–Lagrange equations if and only if

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \rho^i_\alpha y^\alpha, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}, \tag{2.40}$$

for $i \in \{1, \dots, m\}$ and $\alpha \in \{1, \dots, n\}$, where ρ^i_α and $C^\gamma_{\alpha\beta}$ are the structure functions of the Lie algebroid E with respect to the coordinates (x^i) and the local basis $\{e_\alpha\}$ (see section 2.1.1).

In particular, if $\xi \in \Gamma(\mathcal{L}^\tau E)$ is a SODE and

$$i_\xi \omega_L = d^{\mathcal{L}^\tau E} E_L \tag{2.41}$$

then the integral curves of ξ, that is, the integral curves of the vector field $\rho^\tau(\xi)$ are solutions of the Euler–Lagrange equations associated with L.

If the Lagrangian L is regular, that is, ω_L is a nondegenerate section, then there exists a unique solution ξ_L of equation (2.41) and ξ_L is a SODE. In such a case, ξ_L is called *the Euler–Lagrange section associated with* L (for more details, see [29]).

If E is the standard Lie algebroid TM then θ_L (respectively, ω_L and E_L) is the usual Poincaré–Cartan 1-form (respectively, the usual Poincaré–Cartan 2-form and the Lagrangian energy) associated with the Lagrangian function $L : TM \to \mathbb{R}$. In this case, if $L : TM \to \mathbb{R}$ is regular, ξ_L is the Euler–Lagrange vector field.

Remark 2.8. Suppose that (x^i) are coordinates on M and that $\{e_\alpha\}$ is a local basis of sections of E. Denote by (x^i, y^α) the corresponding coordinates on E, by ρ^i_α and $C^\gamma_{\alpha\beta}$ the corresponding structure functions of E and by $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ the local basis of $\Gamma(\mathcal{L}^\tau E)$ considered in remark 2.7.

If $\{\tilde{T}^\alpha, \tilde{V}^\alpha\}$ is the dual basis of $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ then

$$\begin{aligned}
\theta_L &= \frac{\partial L}{\partial y^\alpha}\tilde{T}^\alpha, \\
\omega_L &= \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\tilde{T}^\alpha \wedge \tilde{V}^\beta + \left(\frac{1}{2}\frac{\partial L}{\partial y^\gamma}C^\gamma_{\alpha\beta} - \rho^i_\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta}\right)\tilde{T}^\alpha \wedge \tilde{T}^\beta, \\
E_L &= \frac{\partial L}{\partial y^\alpha} y^\alpha - L.
\end{aligned}$$

Thus, the Lagrangian L is regular if and only if the matrix $(W_{\alpha\beta}) = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)$ is regular. Moreover, if L is regular then the Euler–Lagrange section ξ_L associated with L is given by

$$\xi_L = y^\alpha \tilde{T}_\alpha + W^{\alpha\beta}\left(\rho^i_\beta \frac{\partial L}{\partial x^i} - \rho^i_\gamma y^\gamma \frac{\partial^2 L}{\partial x^i \partial y^\beta} + y^\gamma C^\nu_{\gamma\beta}\frac{\partial L}{\partial y^\nu}\right)\tilde{V}_\alpha, \tag{2.42}$$

where $(W^{\alpha\beta})$ is the inverse matrix of $(W_{\alpha\beta})$.

3. Lie algebroids and Hamiltonian mechanics

3.1. The prolongation of a Lie algebroid over the vector bundle projection of the dual bundle

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $\tau^* : E^* \to M$ be the vector bundle projection of the dual bundle E^* to E.

We consider the prolongation $\mathcal{L}^{\tau^*}E$ of E over τ^*,

$$\mathcal{L}^{\tau^*}E = \{(b, v) \in E \times TE^*/\rho(b) = (T\tau^*)(v)\}.$$

$\mathcal{L}^{\tau^*}E$ is a Lie algebroid over E^* of rank $2n$ with Lie algebroid structure $([\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ defined as follows (see section 2.1.1). If $(f_i'(X_i \circ \tau^*), X')$ and $(s_j'(Y_j \circ \tau^*), Y')$ are two sections of $\mathcal{L}^{\tau^*}E \to E^*$, with $f_i', s_j' \in C^\infty(E^*)$, $X_i, Y_j \in \Gamma(E)$ and $X', Y' \in \mathfrak{X}(E^*)$, then

$$\begin{aligned}[\![(f_i'(X_i \circ \tau^*), X'), (s_j'(Y_j \circ \tau^*), Y')]\!]^{\tau^*} &= (f_i's_j'([\![X_i, Y_j]\!] \circ \tau^*) + X'(s_j')(Y_j \circ \tau^*)\\ &\quad - Y'(f_i')(X_i \circ \tau^*), [X', Y']), \rho^{\tau^*}(f_i'(X_i \circ \tau^*), X') = X'.\end{aligned}$$

Now, if (x^i) are local coordinates on an open subset U of M, $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \to U$ and $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$, then $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ is a basis of sections of the vector bundle $(\tau^{\tau^*})^{-1}((\tau^*)^{-1}(U)) \to (\tau^*)^{-1}(U)$, where $\tau^{\tau^*} : \mathcal{L}^{\tau^*}E \to E^*$ is the vector bundle projection and

$$\tilde{e}_\alpha(a^*) = \left(e_\alpha(\tau^*(a^*)), \rho_\alpha^i \frac{\partial}{\partial x^i}_{|a^*}\right), \qquad \bar{e}_\alpha(a^*) = \left(0, \frac{\partial}{\partial y_\alpha}_{|a^*}\right) \tag{3.1}$$

for $a^* \in (\tau^*)^{-1}(U)$. Here, ρ_α^i are the components of the anchor map with respect to the basis $\{e_\alpha\}$ and (x^i, y_α) are the local coordinates on E^* induced by the local coordinates (x^i) and the basis $\{e^\alpha\}$. In general, if $X = X^\gamma e_\gamma$ is a section of the vector bundle $\tau^{-1}(U) \to U$ then one may consider the sections $\tilde{X}$ and $\bar{X}$ of $(\tau^{\tau^*})^{-1}((\tau^*)^{-1}(U)) \to (\tau^*)^{-1}(U)$ defined by

$$\begin{aligned}\tilde{X}(a^*) &= \left(X(\tau^*(a^*)), \left(\rho_\gamma^i X^\gamma\right) \frac{\partial}{\partial x^i}_{|a^*}\right),\\ \bar{X}(a^*) &= \left(0, X^\gamma(\tau^*(a^*)) \frac{\partial}{\partial y_\gamma}_{|a^*}\right),\end{aligned}$$

for $a^* \in (\tau^*)^{-1}(U)$. Using the local basis $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ one may introduce, in a natural way, local coordinates $(x^i, y_\alpha; z^\alpha, v_\alpha)$ on $\mathcal{L}^{\tau^*}E$. If ω^* is a point of $(\tau^{\tau^*})^{-1}((\tau^*)^{-1}(U))$ then (x^i, y_α) are the coordinates of the point $\tau^{\tau^*}(\omega^*) \in (\tau^*)^{-1}(U)$ and

$$\omega^* = z^\alpha \tilde{e}_\alpha(\tau^{\tau^*}(\omega^*)) + v_\alpha \bar{e}_\alpha(\tau^{\tau^*}(\omega^*)).$$

On the other hand, a direct computation proves that

$$\begin{aligned}&\rho^{\tau^*}(\tilde{e}_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i}, \qquad \rho^{\tau^*}(\bar{e}_\alpha) = \frac{\partial}{\partial y_\alpha},\\ &[\![\tilde{e}_\alpha, \tilde{e}_\beta]\!]^{\tau^*} = \widetilde{[\![e_\alpha, e_\beta]\!]} = C_{\alpha\beta}^\gamma \tilde{e}_\gamma, \qquad [\![\tilde{e}_\alpha, \bar{e}_\beta]\!]^{\tau^*} = [\![\bar{e}_\alpha, \bar{e}_\beta]\!]^{\tau^*} = 0,\end{aligned} \tag{3.2}$$

for all α and β, $C_{\alpha\beta}^\gamma$ being the structure functions of the Lie bracket $[\![\cdot,\cdot]\!]$ with respect to the basis $\{e_\alpha\}$. Thus, if $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ is the dual basis of $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ then

$$\begin{aligned}&d^{\mathcal{L}^{\tau^*}E} f = \rho_\alpha^i \frac{\partial f}{\partial x^i}\tilde{e}^\alpha + \frac{\partial f}{\partial y_\alpha}\bar{e}^\alpha,\\ &d^{\mathcal{L}^{\tau^*}E}\tilde{e}^\gamma = -\frac{1}{2}C_{\alpha\beta}^\gamma \tilde{e}^\alpha \wedge \tilde{e}^\beta, \qquad d^{\mathcal{L}^{\tau^*}E}\bar{e}^\gamma = 0,\end{aligned} \tag{3.3}$$

for $f \in C^\infty(E^*)$.

Remark 3.1. If E is the tangent Lie algebroid TM, then the Lie algebroid $(\mathcal{L}^{\tau^*}E, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ is the standard Lie algebroid $(T(T^*M), [\cdot,\cdot], Id)$.

3.2. The canonical symplectic section of $\mathcal{L}^{\tau^*}E$

Let $(E, [\![,]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $\mathcal{L}^{\tau^*}E$ be the prolongation of E over the vector bundle projection $\tau^* : E^* \to M$. We may introduce a canonical section λ_E of the vector bundle $(\mathcal{L}^{\tau^*}E)^*$ as follows. If $a^* \in E^*$ and (b, v) is a point of the fibre of $\mathcal{L}^{\tau^*}E$ over a^* then

$$\lambda_E(a^*)(b, v) = a^*(b). \tag{3.4}$$

λ_E is called *the Liouville section of* $(\mathcal{L}^{\tau^*}E)^*$.

Now, *the canonical symplectic section* Ω_E is defined by

$$\Omega_E = -d^{\mathcal{L}^{\tau^*}E}\lambda_E. \tag{3.5}$$

We have:

Theorem 3.2. *Ω_E is a symplectic section of the Lie algebroid $(\mathcal{L}^{\tau^*}E, [\![\cdot, \cdot]\!]^{\tau^*}, \rho^{\tau^*})$, that is,*

(i) Ω_E is a nondegenerate 2-section and
(ii) $d^{\mathcal{L}^{\tau^*}E}\Omega_E = 0$.

Proof. It is clear that $d^{\mathcal{L}^{\tau^*}E}\Omega_E = 0$.

On the other hand, if (x^i) are local coordinates on an open subset U of M, $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \to U$, (x^i, y_α) are the corresponding local coordinates of E^* on $(\tau^*)^{-1}(U)$ and $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ is the basis of the vector bundle $(\tau^{\tau^*})^{-1}((\tau^*)^{-1}(U)) \to (\tau^*)^{-1}(U)$ then, using (3.4), it follows that

$$\lambda_E(x^i, y_\alpha) = y_\alpha \tilde{e}^\alpha \tag{3.6}$$

where $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ is the dual basis to $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$. Thus, from (3.3), (3.5) and (3.6), we obtain that

$$\Omega_E = \tilde{e}^\alpha \wedge \bar{e}^\alpha + \tfrac{1}{2}C_{\alpha\beta}^\gamma y_\gamma \tilde{e}^\alpha \wedge \tilde{e}^\beta \tag{3.7}$$

$C_{\alpha\beta}^\gamma$ being the structure functions of the Lie bracket $[\![\cdot, \cdot]\!]$ with respect to the basis $\{e_\alpha\}$.

Therefore, using (3.7), we deduce that Ω_E is a nondegenerate 2-section. □

Remark 3.3. If E is the standard Lie algebroid TM then $\lambda_E = \lambda_{TM}$ (respectively, $\Omega_E = \Omega_{TM}$) is the usual Liouville 1-form (respectively, the canonical symplectic 2-form) on T^*M.

Let $(E, [\![\cdot, \cdot]\!], \rho)$ be a Lie algebroid over a manifold M and γ be a section of the dual bundle E^* to E.

Consider the morphism $((Id, T\gamma \circ \rho), \gamma)$ between the vector bundles E and $\mathcal{L}^{\tau^*}E$

$$\begin{tikzcd} E \arrow[r, "(Id, T\gamma \circ \rho)"] \arrow[d] & \mathcal{L}^{\tau^*}E \arrow[d] \\ M \arrow[r, "\gamma"] & E^* \end{tikzcd}$$

defined by $(Id, T\gamma \circ \rho)(a) = (a, (T_x\gamma)(\rho(a)))$, for $a \in E_x$ and $x \in M$.

Theorem 3.4. *If γ is a section of E^* then the pair $((Id, T\gamma \circ \rho), \gamma)$ is a morphism between the Lie algebroids $(E, [\![\cdot, \cdot]\!], \rho)$ and $(\mathcal{L}^{\tau^*}E, [\![\cdot, \cdot]\!]^{\tau^*}, \rho^{\tau^*})$. Moreover,*

$$((Id, T\gamma \circ \rho), \gamma)^*\lambda_E = \gamma, \qquad ((Id, T\gamma \circ \rho), \gamma)^*(\Omega_E) = -d^E\gamma. \tag{3.8}$$

Proof. Suppose that (x^i) are local coordinates on M, that $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and that

$$\gamma = \gamma_\alpha e^\alpha,$$

with γ_α local real functions on M and $\{e^\alpha\}$ the dual basis to $\{e_\alpha\}$. Denote by $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ the corresponding local basis of $\Gamma(\mathcal{L}^{\tau^*}E)$. Then, using (3.1), it follows that

$$(Id, T\gamma \circ \rho) \circ e_\alpha = \left(\tilde{e}_\alpha + \rho^i_\alpha \frac{\partial \gamma_\nu}{\partial x^i}\bar{e}_\nu\right) \circ \gamma, \tag{3.9}$$

for $\alpha \in \{1, \ldots, n\}$, ρ^i_α being the components of the anchor map with respect to the local coordinates (x^i) and to the basis $\{e_\alpha\}$. Thus, from (2.4),

$$((Id, T\gamma \circ \rho), \gamma)^*(\tilde{e}^\alpha) = e^\alpha, \qquad ((Id, T\gamma \circ \rho), \gamma)^*(\bar{e}^\alpha) = \rho^i_\beta \frac{\partial \gamma_\alpha}{\partial x^i} e^\beta = d^E \gamma_\alpha,$$

where $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ is the dual basis to $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$.

Therefore, from (2.4), (2.5) and (3.3), we obtain that the pair $((Id, T\gamma \circ \rho), \gamma)$ is a morphism between the Lie algebroids $E \to M$ and $\mathcal{L}^{\tau^*}E \to E^*$.

On the other hand, if x is a point of M and $a \in E_x$ then, using (3.4), we have that

$$\{\{((Id, T\gamma \circ \rho), \gamma)^*(\lambda_E)\}(x)\}(a) = \lambda_E(\gamma(x))(a, (T_x\gamma)(\rho(a))) = \gamma(x)(a)$$

that is,

$$((Id, T\gamma \circ \rho), \gamma)^*(\lambda_E) = \gamma.$$

Consequently, from (3.5) and since the pair $((Id, T\gamma \circ \rho), \gamma)$ is a morphism between the Lie algebroids E and $\mathcal{L}^{\tau^*}E$, we deduce that

$$((Id, T\gamma \circ \rho), \gamma)^*(\Omega_E) = -d^E\gamma.$$

□

Remark 3.5. Let $\gamma : M \longrightarrow T^*M$ be a 1-form on a manifold M and λ_{TM} (respectively, Ω_{TM}) be the Liouville 1-form (respectively, the canonical symplectic 2-form) on T^*M. Then, using theorem 3.4 (with $E = TM$), we deduce a well-known result (see, for instance, [1])

$$\gamma^*(\lambda_{TM}) = \gamma, \qquad \gamma^*(\Omega_{TM}) = -d^{TM}\gamma.$$

From theorem 3.4, we also obtain

Corollary 3.6. *If $\gamma \in \Gamma(E^*)$ is a 1-cocycle of E then*

$$((Id, T\gamma \circ \rho), \gamma)^*(\Omega_E) = 0.$$

3.3. The Hamilton equations

Let $H : E^* \to \mathbb{R}$ be a Hamiltonian function. Then, since Ω_E is a symplectic section of $(\mathcal{L}^{\tau^*}E, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ and $d^{\mathcal{L}^{\tau^*}E}H \in \Gamma((\mathcal{L}^{\tau^*}E)^*)$, there exists a unique section $\xi_H \in \Gamma(\mathcal{L}^{\tau^*}E)$ satisfying

$$i_{\xi_H}\Omega_E = d^{\mathcal{L}^{\tau^*}E}H. \tag{3.10}$$

With respect to the local basis $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ of $\Gamma(\mathcal{L}^{\tau^*}E)$, ξ_H is locally expressed as follows:

$$\xi_H = \frac{\partial H}{\partial y_\alpha}\tilde{e}_\alpha - \left(C^\gamma_{\alpha\beta}y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i}\right)\bar{e}_\alpha. \tag{3.11}$$

Thus, the vector field $\rho^{\tau^*}(\xi_H)$ on E^* is given by

$$\rho^{\tau^*}(\xi_H) = \rho^i_\alpha \frac{\partial H}{\partial y_\alpha}\frac{\partial}{\partial x^i} - \left(C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i}\right)\frac{\partial}{\partial y_\alpha}.$$

Therefore, $\rho^{\tau^*}(\xi_H)$ is just *the Hamiltonian vector field of* H with respect to the linear Poisson structure Λ_{E^*} on E^* induced by Lie algebroid structure $([\![\cdot,\cdot]\!],\rho)$, that is,

$$\rho^{\tau^*}(\xi_H) = i_{d^{TE^*}H}\Lambda_{E^*}.$$

Consequently, the integral curves of ξ_H (i.e., the integral curves of the vector field $\rho^{\tau^*}(\xi_H)$) satisfy the Hamilton equations for H,

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \rho^i_\alpha \frac{\partial H}{\partial y_\alpha}, \qquad \frac{\mathrm{d}y_\alpha}{\mathrm{d}t} = -\left(C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i}\right), \tag{3.12}$$

for $i \in \{1,\ldots,m\}$ and $\alpha \in \{1,\ldots,n\}$.

Remark 3.7. If E is the standard Lie algebroid TM then ξ_H is the Hamiltonian vector field of $H : T^*M \to \mathbb{R}$ with respect to the canonical symplectic structure of T^*M and equation (3.12) are the usual Hamilton equations associated with H.

3.4. Complete and vertical lifts

On E^* we have similar concepts of complete and vertical lifts to those in E (see [30]). The existence of a vertical lift is but a consequence of E^* being a vector bundle. Explicitly, given a section $\alpha \in \Gamma(E^*)$ we define the vector field α^v on E^* by

$$\alpha^v(a^*) = \alpha(\tau^*(a^*))^v_{a^*}, \qquad \text{for} \quad a^* \in E^*,$$

where ${}^v_{a^*} : E^*_{\tau^*(a^*)} \longrightarrow T_{a^*}(E^*_{\tau^*(a^*)})$ is the canonical isomorphism between the vector spaces $E^*_{\tau^*(a^*)}$ and $T_{a^*}(E^*_{\tau^*(a^*)})$.

On the other hand, if X is a section of $\tau : E \to M$, there exists a unique vector field X^{*c} on E^*, *the complete lift of* X to E^* satisfying the two following conditions:

(i) X^{*c} is τ^*-projectable on $\rho(X)$ and
(ii) $X^{*c}(\hat{Y}) = \widehat{[X,Y]}$,

for all $Y \in \Gamma(E)$ (see [13]). Here, if X is a section of E then $\hat{X}$ is the linear function $\hat{X} \in C^\infty(E^*)$ defined by

$$\hat{X}(a^*) = a^*(X(\tau^*(a^*))), \qquad \text{for all} \quad a^* \in E^*.$$

We have that (see [13])

$$[X^{*c}, Y^{*c}] = [\![X,Y]\!]^{*c}, \qquad [X^{*c}, \beta^v] = \left(\mathcal{L}^E_X\beta\right)^v, \qquad [\alpha^v, \beta^v] = 0. \tag{3.13}$$

Next, we consider the prolongation $\mathcal{L}^{\tau^*}E$ of E over the projection τ^* (see section 2.1.1). $\mathcal{L}^{\tau^*}E$ is a vector bundle over E^* of rank $2n$. Moreover, we may introduce *the vertical lift* $\alpha^{\mathbf{v}}$ and *the complete lift* $X^{*\mathbf{c}}$ of a section $\alpha \in \Gamma(E^*)$ and a section $X \in \Gamma(E)$ as the sections of $\mathcal{L}^{\tau^*}E \to E^*$ given by

$$\alpha^{\mathbf{v}}(a^*) = (0, \alpha^v(a^*)), \qquad X^{*\mathbf{c}}(a^*) = (X(\tau^*(a^*)), X^{*c}(a^*)) \tag{3.14}$$

for all $a^* \in E^*$. If $\{X_i\}$ is a local basis of $\Gamma(E)$ and $\{\alpha_i\}$ is the dual basis of $\Gamma(E^*)$ then $\left\{\alpha_i^{\mathbf{v}}, X_i^{*\mathbf{c}}\right\}$ is a local basis of $\Gamma(\mathcal{L}^{\tau^*}E)$.

Now, denote by $([\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ the Lie algebroid structure on $\mathcal{L}^{\tau^*}E$ (see section 3.1). It follows that

$$[\![X^{*\mathbf{c}}, Y^{*\mathbf{c}}]\!]^{\tau^*} = [\![X,Y]\!]^{*\mathbf{c}}, \qquad [\![X^{*\mathbf{c}}, \beta^{\mathbf{v}}]\!]^{\tau^*} = \left(\mathcal{L}^E_X\beta\right)^{\mathbf{v}}, \qquad [\![\alpha^{\mathbf{v}}, \beta^{\mathbf{v}}]\!]^{\tau^*} = 0, \tag{3.15}$$

and

$$\begin{aligned} &\rho^{\tau^*}(X^{*\mathbf{c}})(f^v) = (\rho(X)(f))^v, && \rho^{\tau^*}(\alpha^{\mathbf{v}})(f^v) = 0, \\ &\rho^{\tau^*}(X^{*\mathbf{c}})(\hat{Y}) = \widehat{[\![X, Y]\!]}, && \rho^{\tau^*}(\alpha^{\mathbf{v}})(\hat{Y}) = \alpha(Y)^v \end{aligned} \tag{3.16}$$

for $X, Y \in \Gamma(E), \alpha, \beta \in \Gamma(E^*)$ and $f \in C^\infty(M)$. Here, $f^v \in C^\infty(E^*)$ is the basic function on E^* defined by

$$f^v = f \circ \tau^*.$$

Suppose that (x^i) are coordinates on an open subset U of M and that $\{e_\alpha\}$ is a basis of sections of $\tau^{-1}(U) \to U$, and $\{e^\alpha\}$ is the dual basis of sections of $\tau^{*-1}(U) \to U$. Denote by (x^i, y_α) the corresponding coordinates on $\tau^{*-1}(U)$ and by ρ^i_α and $C^\gamma_{\alpha\beta}$ the corresponding structure functions of E. If θ is a section of E^* and on U

$$\theta = \theta_\alpha e^\alpha$$

and X is a section of E and on U

$$X = X^\alpha e_\alpha,$$

then θ^v and X^{*c} are the vector fields on E^* given by

$$\begin{aligned} \theta^v &= \theta_\alpha \frac{\partial}{\partial y_\alpha} \\ X^{*c} &= X^\alpha \rho^i_\alpha \frac{\partial}{\partial x^i} - \left(\rho^i_\alpha \frac{\partial X^\beta}{\partial x^i} y_\beta + C^\gamma_{\alpha\beta} y_\gamma X^\beta \right) \frac{\partial}{\partial y_\alpha}. \end{aligned} \tag{3.17}$$

In particular,

$$e^v_\alpha = \frac{\partial}{\partial y^\alpha}, \qquad e^{*c}_\alpha = \rho^i_\alpha \frac{\partial}{\partial x^i} - C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\beta}. \tag{3.18}$$

In terms of the basis $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ of sections of $\mathcal{L}^{\tau^*}E \to E^*$ we have the local expressions

$$\theta^{\mathbf{v}} = \theta_\alpha \bar{e}_\alpha$$

and

$$X^{*\mathbf{c}} = X^\alpha \tilde{e}_\alpha - \left(\rho^i_\alpha \frac{\partial X^\beta}{\partial x^i} y_\beta + C^\gamma_{\alpha\beta} y_\gamma X^\beta \right) \bar{e}_\alpha.$$

Remark 3.8. If E is the standard Lie algebroid TM, then $\mathcal{L}^{\tau^*}E = T(T^*M)$ and the vertical and complete lifts of sections are the usual vertical and complete lifts.

The following properties relate vertical and complete lifts with the Liouville 1-section and the canonical symplectic 2-section.

Proposition 3.9. *If X, Y are sections of E, β, θ are sections of E^* and λ_E is the Liouville 1-section, then*

$$\lambda_E(\theta^{\mathbf{v}}) = 0 \qquad \text{and} \qquad \lambda_E(X^{*\mathbf{c}}) = \hat{X}. \tag{3.19}$$

If Ω_E is the canonical symplectic 2-section then

$$\Omega_E(\beta^{\mathbf{v}}, \theta^{\mathbf{v}}) = 0, \qquad \Omega_E(X^{*\mathbf{c}}, \theta^{\mathbf{v}}) = \theta(X)^v \qquad \text{and} \qquad \Omega_E(X^{*\mathbf{c}}, Y^{*\mathbf{c}}) = -\widehat{[\![X, Y]\!]}. \tag{3.20}$$

Proof. Indeed, for every $a^* \in E^*$,

$$\lambda_E(\theta^{\mathbf{v}})(a^*) = a^*(pr_1(\theta^{\mathbf{v}}(a^*))) = a^*(0_{\tau^*(a^*)}) = 0,$$

and

$$\lambda_E(X^{*\mathbf{c}})(a^*) = a^*(pr_1(X^{*\mathbf{c}}(a^*))) = a^*(X(\tau^*(a^*))) = \hat{X}(a^*),$$

which proves (3.19).

For (3.20) we take into account (3.19) and the definition of the differential d, so that

$$\begin{aligned}
\Omega_E(\theta^{\mathbf{v}}, \beta^{\mathbf{v}}) &= \rho^{\tau^*}(\beta^{\mathbf{v}})(\lambda_E(\theta^{\mathbf{v}})) - \rho^{\tau^*}(\theta^{\mathbf{v}})(\lambda_E(\beta^{\mathbf{v}})) + \lambda_E([\![\theta^{\mathbf{v}}, \beta^{\mathbf{v}}]\!]^{\tau^*}) = 0,\\
\Omega_E(X^{*\mathbf{c}}, \beta^{\mathbf{v}}) &= \rho^{\tau^*}(\beta^{\mathbf{v}})(\lambda_E(X^{*\mathbf{c}})) - \rho^{\tau^*}(X^{*\mathbf{c}})(\lambda_E(\beta^{\mathbf{v}})) + \lambda_E([\![X^{*\mathbf{c}}, \beta^{\mathbf{v}}]\!]^{\tau^*})\\
&= \rho^{\tau^*}(\beta^{\mathbf{v}})\hat{X} = \beta(X)^v,\\
\Omega_E(X^{*\mathbf{c}}, Y^{*\mathbf{c}}) &= \rho^{\tau^*}(Y^{*\mathbf{c}})(\lambda_E(X^{*\mathbf{c}})) - \rho^{\tau^*}(X^{*\mathbf{c}})(\lambda_E(Y^{*\mathbf{c}})) + \lambda_E([\![X^{*\mathbf{c}}, Y^{*\mathbf{c}}]\!]^{\tau^*})\\
&= \rho^{\tau^*}(Y^{*\mathbf{c}})\hat{X} - \rho^{\tau^*}(X^{*\mathbf{c}})\hat{Y} + \widehat{[\![X, Y]\!]} = -\widehat{[\![X, Y]\!]},
\end{aligned}$$

where we have used (3.15) and (3.16). □

Noether's theorem has a direct generalization to the theory of dynamical Hamiltonian systems on Lie algebroids. By an infinitesimal symmetry of a section X of a Lie algebroid we mean another section Y which commutes with X, that is, $[\![X, Y]\!] = 0$.

Theorem 3.10. *Let $H \in C^\infty(E^*)$ be a Hamiltonian function and ξ_H be the corresponding Hamiltonian section. If $X \in \Gamma(E)$ is a section of E such that $\rho^{\tau^*}(X^{*\mathbf{c}})H = 0$ then $X^{*\mathbf{c}}$ is a symmetry of ξ_H and the function $\hat{X}$ is a constant of the motion, that is $\rho^{\tau^*}(\xi_H)\hat{X} = 0$.*

Proof. Using (3.10) and since $\rho^{\tau^*}(X^{*\mathbf{c}})(H) = 0$, it follows that

$$\mathcal{L}^{\mathcal{L}^{\tau^*}E}_{X^{*\mathbf{c}}}\big(i_{\xi_H}\Omega_E\big) = d^{\mathcal{L}^{\tau^*}E}\big(\mathcal{L}^{\mathcal{L}^{\tau^*}E}_{X^{*\mathbf{c}}}H\big) = 0. \tag{3.21}$$

On the other hand, from (3.16) and (3.20), we obtain that

$$i_{X^{*\mathbf{c}}}\Omega_E = -d^{\mathcal{L}^{\tau^*}E}\hat{X} \tag{3.22}$$

and thus

$$\mathcal{L}^{\mathcal{L}^{\tau^*}E}_{X^{*\mathbf{c}}}\Omega_E = 0.$$

Therefore, using (3.21), we deduce that

$$i_{[\![\xi_H, X^{*\mathbf{c}}]\!]^{\tau^*}}\Omega_E = 0,$$

which implies that $[\![\xi_H, X^{*\mathbf{c}}]\!]^{\tau^*} = 0$, that is, $X^{*\mathbf{c}}$ is a symmetry of ξ_H.

In addition, from (3.10) and (3.22), we conclude that

$$0 = \Omega_E(\xi_H, X^{*\mathbf{c}}) = \mathcal{L}^{\mathcal{L}^{\tau^*}E}_{\xi_H}\hat{X} = \rho^{\tau^*}(\xi_H)(\hat{X}).$$

□

3.5. Poisson bracket

Let E be a Lie algebroid over M. Then, as we know, E^* admits a linear Poisson structure Λ_{E^*} with linear Poisson bracket $\{\,,\}_{E^*}$ (see section 2.1).

Next, we will prove that the Poisson bracket $\{\,,\}_{E^*}$ can also be defined in terms of the canonical symplectic 2-section Ω_E.

Proposition 3.11. *Let $F, G \in C^\infty(E^*)$ be functions on E^* and ξ_F, ξ_G be the corresponding Hamiltonian sections. Then, the Poisson bracket of F and G is*

$$\{F, G\}_{E^*} = -\Omega_E(\xi_F, \xi_G).$$

Proof. We will see that the Hamiltonian section defined by a basic function f^v is $-(d^E f)^{\mathbf{v}}$, the vertical lift of $-d^E f \in \Gamma(E^*)$, and that the Hamiltonian section defined by a linear function $\hat{X}$ is $X^{*\mathbf{c}}$, the complete lift of $X \in \Gamma(E)$.

Indeed, using (3.16) and (3.20), we deduce that

$$\Omega_E(\xi_{f^v}, \theta^{\mathbf{v}}) = \left(d^{\mathcal{L}^{\tau^*}E} f^v\right)(\theta^{\mathbf{v}}) = 0 = \Omega_E((-d^E f)^{\mathbf{v}}, \theta^{\mathbf{v}}),$$
$$\Omega_E(\xi_{f^v}, Y^{*\mathbf{c}}) = \left(d^{\mathcal{L}^{\tau^*}E} f^v\right)(Y^{*\mathbf{c}}) = (d^E f(Y))^v = \Omega_E((-d^E f)^{\mathbf{v}}, Y^{*\mathbf{c}}),$$

and similarly

$$\Omega_E(\xi_{\hat{X}}, \theta^{\mathbf{v}}) = \left(d^{\mathcal{L}^{\tau^*}E} \hat{X}\right)(\theta^{\mathbf{v}}) = \rho^{\tau^*}(\theta^{\mathbf{v}})\hat{X} = (\theta(X))^v = \Omega_E(X^{*\mathbf{c}}, \theta^{\mathbf{v}}),$$
$$\Omega_E(\xi_{\hat{X}}, Y^{*\mathbf{c}}) = \left(d^{\mathcal{L}^{\tau^*}E} \hat{X}\right)(Y^{*\mathbf{c}}) = \rho^{\tau^*}(Y^{*\mathbf{c}})\hat{X} = -\widehat{[\![X, Y]\!]} = \Omega_E(X^{*\mathbf{c}}, Y^{*\mathbf{c}}),$$

for every $Y \in \Gamma(E)$ and every $\theta \in \Gamma(E^*)$. The proof follows using (2.9), (3.20) and by taking into account that complete and vertical lifts generate $\Gamma(\mathcal{L}^{\tau^*}E)$. □

3.6. The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms

Let $L : E \to \mathbb{R}$ be a Lagrangian function and $\theta_L \in \Gamma((\mathcal{L}^\tau E)^*)$ be the Poincaré–Cartan 1-section associated with L.

We introduce *the Legendre transformation associated with* L as the smooth map $Leg_L : E \to E^*$ defined by

$$Leg_L(a)(b) = \theta_L(a)(z), \tag{3.23}$$

for $a, b \in E_x$, where E_x is the fibre of E over the point $x \in M$ and z is a point in the fibre of $\mathcal{L}^\tau E$ over the point a such that

$$pr_1(z) = b,$$

$pr_1 : \mathcal{L}^\tau E \to E$ being the restriction to $\mathcal{L}^\tau E$ of the first canonical projection $pr_1 : E \times TE \to E$.

The map Leg_L is well defined and its local expression in fibred coordinates on E and E^* is

$$Leg_L(x^i, y^\alpha) = \left(x^i, \frac{\partial L}{\partial y^\alpha}\right). \tag{3.24}$$

The Legendre transformation induces a map $\mathcal{L}Leg_L : \mathcal{L}^\tau E \to \mathcal{L}^{\tau^*}E$ defined by

$$(\mathcal{L}Leg_L)(b, X_a) = (b, (T_a Leg_L)(X_a)), \tag{3.25}$$

for $a, b \in E$ and $(b, X_a) \in (\mathcal{L}^\tau E)_a \subseteq E_{\tau(a)} \times T_a E$, where $TLeg_L : TE \to TE^*$ is the tangent map of Leg_L. Note that $\tau^* \circ Leg_L = \tau$ and thus $\mathcal{L}Leg_L$ is well-defined.

Using (3.24), we deduce that the local expression of $\mathcal{L}Leg_L$ in the coordinates of $\mathcal{L}^\tau E$ and $\mathcal{L}^{\tau^*}E$ (see sections 2.2.1 and 3.1) is

$$\mathcal{L}Leg_L(x^i, y^\alpha; z^\alpha, v^\alpha) = \left(x^i, \frac{\partial L}{\partial y^\alpha}; z^\alpha, \rho^i_\beta z^\beta \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + v^\beta \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right). \tag{3.26}$$

Theorem 3.12. *The pair $(\mathcal{L}Leg_L, Leg_L)$ is a morphism between the Lie algebroids $(\mathcal{L}^\tau E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ and $(\mathcal{L}^{\tau^*}E, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$. Moreover, if θ_L and ω_L (respectively, λ_E and Ω_E) are the Poincaré–Cartan 1-section and 2-section associated with L (respectively, the Liouville 1-section and the canonical symplectic section on $\mathcal{L}^{\tau^*}E$) then*

$$(\mathcal{L}Leg_L, Leg_L)^*(\lambda_E) = \theta_L, \qquad (\mathcal{L}Leg_L, Leg_L)^*(\Omega_E) = \omega_L. \tag{3.27}$$

Proof. Suppose that (x^i) are local coordinates on M, that $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and denote by $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ (respectively, $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$) the corresponding local basis of $\Gamma(\mathcal{L}^\tau E)$ (respectively, $\Gamma(\mathcal{L}^{\tau^*} E)$). Then, using (2.36) and (3.26), we deduce that

$$(\mathcal{L}Leg_L, Leg_L)^*(\tilde{e}^\gamma) = \tilde{T}^\gamma, \qquad (\mathcal{L}Leg_L, Leg_L)^*(\bar{e}^\gamma) = d^{\mathcal{L}^\tau E}\left(\frac{\partial L}{\partial y^\gamma}\right), \qquad \text{for all} \quad \gamma.$$

Thus, from (2.36) and (3.3), we conclude that

$$(\mathcal{L}Leg_L, Leg_L)^*\big(d^{\mathcal{L}^{\tau^*} E} f'\big) = d^{\mathcal{L}^\tau E}(f' \circ Leg_L), \tag{3.28}$$

$$(\mathcal{L}Leg_L, Leg_L)^*\big(d^{\mathcal{L}^{\tau^*} E} \tilde{e}^\gamma\big) = d^{\mathcal{L}^\tau E}((\mathcal{L}Leg_L, Leg_L)^*\tilde{e}^\gamma), \tag{3.29}$$

$$(\mathcal{L}Leg_L, Leg_L)^*\big(d^{\mathcal{L}^{\tau^*} E} \bar{e}^\gamma\big) = d^{\mathcal{L}^\tau E}((\mathcal{L}Leg_L, Leg_L)^*\bar{e}^\gamma), \tag{3.30}$$

for all $f' \in C^\infty(E^*)$ and for all γ. Consequently, the pair $(\mathcal{L}Leg_L, Leg_L)$ is a Lie algebroid morphism. This result also follows using proposition 1.8 in [17].

Now, from (3.4) and (3.23), we obtain that

$$(\mathcal{L}Leg_L, Leg_L)^*(\lambda_E) = \theta_L.$$

Thus, using (2.38), (3.5) and the first part of the theorem, we deduce that

$$(\mathcal{L}Leg_L, Leg_L)^*(\Omega_E) = \omega_L.$$

□

We also may prove the following result.

Proposition 3.13. *The Lagrangian L is regular if and only if the Legendre transformation $Leg_L : E \to E^*$ is a local diffeomorphism.*

Proof. L is regular if and only if the matrix $\left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)$ is regular. Therefore, using (3.24), we deduce the result. □

Next, we will assume that L is *hyperregular*, that is, Leg_L is a global diffeomorphism. Then, from (3.25) and theorem 3.12, we conclude that the pair $(\mathcal{L}Leg_L, Leg_L)$ is a Lie algebroid isomorphism. Moreover, we may consider the Hamiltonian function $H : E^* \to \mathbb{R}$ defined by

$$H = E_L \circ Leg_L^{-1},$$

where $E_L : E \to \mathbb{R}$ is the Lagrangian energy associated with L given by (2.39). The Hamiltonian section $\xi_H \in \Gamma(\mathcal{L}^{\tau^*} E)$ is characterized by the condition

$$i_{\xi_H} \Omega_E = d^{\mathcal{L}^{\tau^*} E} H \tag{3.31}$$

and we have the following.

Theorem 3.14. *If the Lagrangian L is hyperregular then the Euler–Lagrange section ξ_L associated with L and the Hamiltonian section ξ_H are $(\mathcal{L}Leg_L, Leg_L)$-related, that is,*

$$\xi_H \circ Leg_L = \mathcal{L}Leg_L \circ \xi_L. \tag{3.32}$$

Moreover, if $\gamma : I \to E$ is a solution of the Euler–Lagrange equations associated with L, then $\mu = Leg_L \circ \gamma : I \to E^*$ is a solution of the Hamilton equations associated with H and, conversely, if $\mu : I \to E^*$ is a solution of the Hamilton equations for H then $\gamma = Leg_L^{-1} \circ \mu$ is a solution of the Euler–Lagrange equations for L.

Proof. From (3.27), (3.28) and (3.31), we obtain that (3.32) holds. Now, using (3.32) and theorem 3.12, we deduce the second part of the theorem. □

Remark 3.15. If E is the standard Lie algebroid TM then $Leg_L : TM \to T^*M$ is the usual Legendre transformation associated with $L : TM \to \mathbb{R}$ and theorem 3.14 gives the equivalence between the Lagrangian and Hamiltonian formalisms in classical mechanics.

3.7. The Hamilton–Jacobi equation

The aim of this section is to prove the following result.

Theorem 3.16. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $([\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ be the Lie algebroid structure on $\mathcal{L}^{\tau^*}E$. Let $H : E^* \longrightarrow \mathbb{R}$ be a Hamiltonian function and $\xi_H \in \Gamma(\mathcal{L}^{\tau^*}E)$ be the corresponding Hamiltonian section. Let $\alpha \in \Gamma(E^*)$ be a 1-cocycle, $d^E\alpha = 0$, and denote by $\sigma \in \Gamma(E)$ the section $\sigma = pr_1 \circ \xi_H \circ \alpha$. Then, the following two conditions are equivalent:*

(i) For every curve $t \to c(t)$ in M satisfying

$$\rho(\sigma)(c(t)) = \dot{c}(t), \qquad \textit{for all} \quad t, \tag{3.33}$$

the curve $t \to \alpha(c(t))$ on E^* satisfies the Hamilton equations for H.

(ii) α satisfies the Hamilton–Jacobi equation $d^E(H \circ \alpha) = 0$, that is, the function $H \circ \alpha : M \longrightarrow \mathbb{R}$ is constant on the leaves of the Lie algebroid foliation associated with E.

Proof. For a curve $c : I = (-\epsilon, \epsilon) \subset \mathbb{R} \longrightarrow M$ on the base we define the curves $\mu : I \longrightarrow E^*$ and $\gamma : I \longrightarrow E$ by

$$\mu(t) = \alpha(c(t)) \qquad \text{and} \qquad \gamma(t) = \sigma(c(t)).$$

Since α and σ are sections, it follows that both curves project to c. We consider the curve $v = (\gamma, \dot{\mu})$ in $E \times TE^*$ and note the following important facts about v:

- $v(t)$ is in $\mathcal{L}^{\tau^*}E$, for every $t \in I$, if and only if c satisfies (3.33). Indeed $\rho \circ \gamma = \rho \circ \sigma \circ c$ while $T\tau^* \circ \dot{\mu} = \dot{c}$.
- In such a case, μ is a solution of the Hamilton equations if and only if $v(t) = \xi_H(\mu(t))$, for every $t \in I$. Indeed, the first components coincide $pr_1(v(t)) = \gamma(t)$ and $pr_1(\xi_H(\mu(t))) = pr_1(\xi_H(\alpha(c(t)))) = \sigma(c(t)) = \gamma(t)$, and the equality of the second components is just $\dot{\mu}(t) = \rho^{\tau^*}(\xi_H(\mu(t)))$.

We also consider the map $\Phi_\alpha : E \longrightarrow \mathcal{L}^{\tau^*}E$ given by $\Phi_\alpha = (Id, T\alpha \circ \rho)$, and we recall that $\Omega_E(\alpha(x))(\Phi_\alpha(a), \Phi_\alpha(b)) = 0$, for all $a, b \in E_x$, because of corollary 3.6.

$[(ii) \Rightarrow (i)]$ Assume that c satisfies (3.33), so that $v(t)$ is a curve in $\mathcal{L}^{\tau^*}E$. We have to prove that $v(t)$ equals $\xi_H(\mu(t))$, for every $t \in I$.

The difference $d(t) = v(t) - \xi_H(\mu(t))$ is vertical, that is, $pr_1(d(t)) = 0$, for all t (note that $pr_1(v(t)) = pr_1(\xi_H(\mu(t))) = \gamma(t)$, for all t). Therefore, we have that $\Omega_E(\mu(t))(d(t), \eta(t)) = 0$, for every vertical curve $t \to \eta(t)$ (see (3.7)).

Let $a : I \longrightarrow E$ be any curve on E over c (that is, $\tau \circ a = c$) and consider its image under Φ_α, that is $\zeta(t) = \Phi_\alpha(a(t)) = (a(t), T\alpha(\rho(a(t))))$. Since $v(t) = \Phi_\alpha(\gamma(t))$ is also in the image of Φ_α we have that $\Omega_E(\mu(t))(v(t), \zeta(t)) = 0$. Thus

$$\begin{aligned}\Omega_E(\mu(t))(d(t), \zeta(t)) &= -\Omega_E(\mu(t))(\xi_H(\mu(t)), \zeta(t)) = -\left\langle d^{\mathcal{L}^{\tau^*}E}H, \zeta(t)\right\rangle \\ &= -T\alpha(\rho(a(t)))H = -\langle d^E(H \circ \alpha), a(t)\rangle\end{aligned}$$

which vanishes because $d^E(H \circ \alpha) = 0$.

Since any element in $(\mathcal{L}^{\tau^*}E)_{\alpha(x)}$, with $x \in M$, can be obtained as a sum of an element in the image of Φ and a vertical, we conclude that $\Omega_E(\mu(t))(d(t), \eta(t)) = \Omega_E(\alpha(c(t)))(d(t), \eta(t)) = 0$ for every curve $t \to \eta(t)$, which amounts to $d(t) = 0$.

$[(i) \Rightarrow (ii)]$ Suppose that x is a point of M and that $b \in E_x$. We will show that

$$\langle d^E(H \circ \alpha), b\rangle = 0. \tag{3.34}$$

Let $c : I = (-\epsilon, \epsilon) \to M$ be the integral curve of $\rho(\sigma)$ such that $c(0) = x$. It follows that c satisfies (3.33). Let $\mu = \alpha \circ c, \gamma = \sigma \circ c$ and $v = (\gamma, \dot{\mu})$ as above. Since μ satisfies the Hamilton equations, we have that $v(t) = \xi_H(\mu(t))$ for all t. As above we take any curve $t \to a(t)$ in E over c such that $a(0) = b$. Since $v(t) = \Phi_\alpha(\gamma(t))$ we have that

$$
\begin{aligned}
0 &= \Omega_E(\mu(t))(\Phi_\alpha(\gamma(t)), \Phi_\alpha(a(t))) = \Omega_E(\mu(t))(v(t), \Phi_\alpha(a(t))) \\
&= \Omega_E(\mu(t))(\xi_H(\mu(t)), \Phi_\alpha(a(t))) = \left\langle d^{\mathcal{L}^{\tau^*}E} H, \Phi_\alpha(a(t))\right\rangle = \langle d^E(H \circ \alpha), a(t)\rangle.
\end{aligned}
$$

In particular, at $t = 0$ we have that $\langle d^E(H \circ \alpha), b\rangle = 0$. $\square$

Remark 3.17. Obviously, we can consider as a cocycle α a 1-coboundary $\alpha = d^E S$, for some function S on M. Nevertheless, it should be noted that on a Lie algebroid there exist, in general, 1-cocycles that are not locally 1-coboundaries.

Remark 3.18. If we apply theorem 3.16 to the particular case when E is the standard Lie algebroid TM then we directly deduce a well-known result (see theorem 5.2.4 in [1]).

Let $L : E \longrightarrow \mathbb{R}$ be a hyperregular Lagrangian and $Leg_L : E \longrightarrow E^*$ be the Legendre transformation associated with L. Denote by $H : E^* \longrightarrow \mathbb{R}$ the corresponding Hamiltonian function, that is,

$$H = E_L \circ Leg_L^{-1},$$

E_L being the energy for L.

Now, suppose that $\alpha = d^E S$, for $S : M \longrightarrow \mathbb{R}$ a function on M, is a solution of the Hamilton–Jacobi equation $d^E(H \circ d^E S) = 0$ and that $\mu : I = (-\varepsilon, \varepsilon) \longrightarrow E^*$ is a solution of the Hamilton equations for H such that $\mu(0) = d^E S(x)$, x being a point of M. If $c : I \longrightarrow M$ is the projection of the curve μ to M (that is, $c = \tau^* \circ \mu$) then, from theorem 3.16, we deduce that

$$\mu = d^E S \circ c.$$

On the other hand, the curve $\gamma : I \longrightarrow E$ given by

$$\gamma = Leg_L^{-1} \circ \mu$$

is a solution of the Euler–Lagrange equations associated with L (see theorem 3.14). Moreover, since $c = \tau \circ \gamma$ and γ is admissible, it follows that

$$\dot{c}(t) = \rho(\gamma(t)), \qquad \text{for all} \quad t.$$

Thus, if $\mathcal{F}^E$ is the Lie algebroid foliation associated with E, we have that

$$c(I) \subseteq \mathcal{F}_x^E,$$

where $\mathcal{F}_x^E$ is the leaf of $\mathcal{F}^E$ over the point x.

In addition, $H \circ d^E S$ is constant on $\mathcal{F}_x^E$. We will show next that in the case that the constant is 0 the function S is but the action.

Proposition 3.19. *Let $\alpha = d^E S$ be a solution of the Hamilton–Jacobi equation such that*

$$H \circ d^E S = 0 \qquad \text{on} \quad \mathcal{F}_x^E.$$

Then,

$$\frac{\mathrm{d}(S \circ c)}{\mathrm{d}t} = L \circ \gamma.$$

Proof. Since $Leg_L \circ \gamma = \mu$, we have that $(\Delta L) \circ \gamma = \langle \mu, \gamma \rangle$, and from $E_L = \Delta L - L$ we get $L \circ \gamma = \langle \mu, \gamma \rangle - H \circ \mu$. Moreover,

$$\langle \mu(t), \gamma(t) \rangle = \langle d^E S(c(t)), \gamma(t) \rangle = \rho(\gamma(t))S = \dot{c}(t)S = \frac{\mathrm{d}}{\mathrm{d}t}(S \circ c).$$

In our case, c is a curve on the leaf $\mathcal{F}_x^E$ and $\mu = d^E S \circ c$, so that $H \circ \mu = 0$. Therefore, we immediately get $L \circ \gamma = \frac{\mathrm{d}}{\mathrm{d}t}(S \circ c)$. □

In particular, if we consider a curve $c : [0, T] \longrightarrow M$ such that $c(T) = y$ then

$$S(y) = \int_0^T L(\gamma(t))\, \mathrm{d}t,$$

where we have put $S(x) = 0$, since S is undetermined by a constant.

4. The canonical involution for Lie algebroids

Let M be a smooth manifold, TM be its tangent bundle and $\tau_M : TM \to M$ be the canonical projection. Then, the tangent bundle to $TM, T(T(M))$, admits two vector bundle structures. The vector bundle projection of the first structure (respectively, the second structure) is the canonical projection $\tau_{TM} : T(T(M)) \to TM$ (respectively, the tangent map $T(\tau_M) : T(T(M)) \to TM$ to $\tau_M : TM \to M$). Moreover, the canonical involution $\sigma_{TM} : T(T(M)) \to T(T(M))$ is an isomorphism between the vector bundles $\tau_{TM} : T(T(M)) \to TM$ and $T(\tau_M) : T(T(M)) \to TM$. We recall that if (x^i) are local coordinates on M and (x^i, y^i) (respectively, $(x^i, y^i; \dot{x}^i, \dot{y}^i)$) are the corresponding fibred coordinates on TM (respectively, $T(T(M))$) then the local expression of σ_{TM} is

$$\sigma_{TM}(x^i, y^i; \dot{x}^i, \dot{y}^i) = (x^i, \dot{x}^i; y^i, \dot{y}^i)$$

(for more details, see [12, 20]).

Now, suppose that $(E, [\![\cdot, \cdot]\!], \rho)$ is a Lie algebroid of rank n over a manifold M of dimension m and that $\tau : E \to M$ is the vector bundle projection.

Lemma 4.1. *Let (b, v) be a point in $(\mathcal{L}^\tau E)_a$, with $a \in E_x$, that is, $\tau^\tau(b, v) = a$. Then, there exists one and only one tangent vector $\bar{v} \in T_b E$ such that:*

(i) $\bar{v}(f^v) = (d^E f)(x)(a)$, *and*
(ii) $\bar{v}(\hat{\theta}) = v(\hat{\theta}) + (d^E \theta)(x)(a, b)$

for all $f \in C^\infty(M)$ *and* $\theta \in \Gamma(E^*)$.

Proof. Conditions (i) and (ii) determine a vector on E provided that they are compatible. To prove compatibility, we take $f \in C^\infty(M)$ and $\theta \in \Gamma(E^*)$. Then,

$$\bar{v}(\widehat{f\theta}) = \bar{v}(f^v \hat{\theta}) = (\bar{v}(f^v))\hat{\theta}(b) + f^v(b)(\bar{v}(\hat{\theta})) = (d^E f)(x)(a)\theta(x)(b) + f(x)\bar{v}(\hat{\theta}),$$

where $x = \tau(a) = \tau(b)$, and on the other hand,

$$\begin{aligned} v(\widehat{f\theta}) + (d^E(f\theta))(x)(a, b) &= (v(f^v))\hat{\theta}(a) + f^v(a)v(\hat{\theta}) + (d^E f \wedge \theta)(x)(a, b) \\ &\quad + f(x)(d^E\theta)(x)(a, b) \\ &= f(x)[v(\hat{\theta}) + (d^E\theta)(x)(a, b)] + (d^E f)(x)(a)\theta(x)(b), \end{aligned}$$

which are equal. □

The tangent vector $\bar{v}$ in the above lemma 4.1 projects to $\rho(a)$ since

$$((T_b\tau)(\bar{v}))f = \bar{v}(f \circ \tau) = \bar{v}(f^v) = (d^E f)(x)(a) = \rho(a)f,$$

for all functions $f \in C^\infty(M)$, and thus $(T_b\tau)(\bar{v}) = \rho(a)$. It follows that $(a, \bar{v})$ is an element of $(\mathcal{L}^\tau E)_b$, and we have defined a map from $\mathcal{L}^\tau E$ to $\mathcal{L}^\tau E$.

Theorem 4.2. *The map $\sigma_E : \mathcal{L}^\tau E \longrightarrow \mathcal{L}^\tau E$ given by*

$$\sigma_E(b, v) = (a, \bar{v}),$$

where $\bar{v}$ is the tangent vector whose existence is ensured by lemma 4.1 is a smooth involution interwining the projections τ^τ and pr_1, that is

(i) $\sigma_E^2 = Id$, and
(ii) $pr_1 \circ \sigma_E = \tau^\tau$.

Proof. We will find its coordinate expression. Suppose that (x^i) are local coordinates on an open subset U of M, that $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \to U$ and that $(x^i, y^\alpha; z^\alpha, v^\alpha)$ are the coordinates of (b, v), so that (x^i, y^α) are the coordinates of a, (x^i, z^α) are the coordinates of b and $v = \rho^i_\alpha z^\alpha \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}$. Then $\bar{v} = \rho^i_\alpha y^\alpha \frac{\partial}{\partial x^i} + \bar{v}^\alpha \frac{\partial}{\partial y^\alpha}$ where we have to determine $\bar{v}^\alpha$. Denote by $\{e^\alpha\}$ the dual basis of $\{e_\alpha\}$. Applying $\bar{v}$ to $y^\alpha = \widehat{e^\alpha}$ and taking into account the definition of $\bar{v}$ we get

$$\bar{v}^\alpha = \bar{v}(y^\alpha) = v(y^\alpha) + (d^E e^\alpha)(x)(a, b) = v^\alpha - C^\alpha_{\beta\gamma} y^\beta z^\gamma.$$

Therefore, in coordinates

$$\sigma_E(x^i, y^\alpha; z^\alpha, v^\alpha) = \left(x^i, z^\alpha; y^\alpha, v^\alpha + C^\alpha_{\beta\gamma} z^\beta y^\gamma\right) \tag{4.1}$$

which proves that σ_E is smooth.

Moreover $\tau^\tau(b, v) = a$ and $pr_1(\sigma_E(b, v)) = pr_1(a, \bar{v}) = a$. Thus, (ii) holds. Finally, σ_E is an involution. In fact, if $(b, v) \in (\mathcal{L}^\tau E)_a$ then

$$\sigma_E^2(b, v) = \sigma_E(a, \bar{v}) = (b, \bar{\bar{v}})$$

and $\bar{\bar{v}} = v$ because both project to $\rho(b)$ and over linear functions

$$\bar{\bar{v}}(\hat{\theta}) = \bar{v}(\hat{\theta}) + (d^E\theta)(x)(b, a) = v(\hat{\theta}) + (d^E\theta)(x)(a, b) + (d^E\theta)(x)(b, a) = v(\hat{\theta}),$$

which concludes the proof. □

Definition 4.3. *The map σ_E will be called the canonical involution associated with the Lie algebroid E.*

If E is the standard Lie algebroid TM then $\sigma_E \equiv \sigma_{TM}$ is the usual canonical involution $\sigma_{TM} : T(TM) \to T(TM)$. In this case the canonical involution has the following interpretation (see [45]). Let $\chi : \mathbb{R}^2 \longrightarrow M$ be a map locally defined in a neighbourhood of the origin in $\mathbb{R}^2$. We can consider χ as an one-parameter family of curves in two alternative ways. If (s, t) are the coordinates in $\mathbb{R}^2$, then we can consider the curve $\chi_t : s \mapsto \chi(s, t)$, for fixed t. If we take the tangent vector at $s = 0$ for every t we get a curve $A(t) = \frac{\mathrm{d}\chi_t}{\mathrm{d}s}\big|_{s=0} = \frac{\partial \chi}{\partial s}(0, t)$ in TM whose tangent vector at $t = 0$ is

$$v = \dot{A}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\chi_t}{\mathrm{d}s}\Big|_{s=0}\Big|_{t=0} = \frac{\partial}{\partial t}\frac{\partial \chi}{\partial s}(0, 0).$$

v is a tangent vector to TM at $a = A(0)$. On the other hand, we can consider the curve $\chi^s : t \mapsto \chi(s, t)$, for fixed s. If we take the tangent vector at $t = 0$ for every s we get a curve $B(s) = \frac{\mathrm{d}\chi^s}{\mathrm{d}t}\big|_{t=0} = \frac{\partial \chi}{\partial t}(s, 0)$ in TM whose tangent vector at $s = 0$ is

$$\bar{v} = \dot{B}(0) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\chi^s}{\mathrm{d}t}\Big|_{t=0}\Big|_{s=0} = \frac{\partial}{\partial s}\frac{\partial \chi}{\partial t}(0, 0).$$

$\bar{v}$ is a tangent vector to TM at $b = B(0)$. We have that $v \in T_a(TM)$ projects to b and $\bar{v} \in T_b(TM)$ projects to a. The canonical involution on TM maps one of these vectors into the other one, that is, $\sigma_{TM}(v) = \bar{v}$. Note that in terms of the tangent map $T\chi$ the curves A and B are given by

$$A(t) = T\chi\left(\frac{\partial}{\partial s}\Big|_{(0,t)}\right) \qquad \text{and} \qquad B(s) = T\chi\left(\frac{\partial}{\partial t}\Big|_{(s,0)}\right).$$

We look for a similar description in the case of an arbitrary Lie algebroid. For that we consider a morphism $\Phi : T\mathbb{R}^2 \longrightarrow E$, locally defined in $\tau_{\mathbb{R}^2}^{-1}(\mathcal{O})$ for some open neighbourhood $\mathcal{O}$ of the origin, from the standard Lie algebroid $\tau_{\mathbb{R}^2} : T\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ to our Lie algebroid $\tau : E \longrightarrow M$ and denote by χ the base map $\chi : \mathbb{R}^2 \longrightarrow M$, locally defined in $\mathcal{O}$. (The map Φ plays the role of $T\chi$ in the standard case.)

The vector fields $\left\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\}$ are a basis of $\Gamma(T\mathbb{R}^2)$ with dual basis $\{\mathrm{d}s, \mathrm{d}t\}$. Thus we have the curves $A : \mathbb{R} \longrightarrow E$ and $B : \mathbb{R} \longrightarrow E$ given by

$$A(t) = \Phi\left(\frac{\partial}{\partial s}\Big|_{(0,t)}\right) \qquad \text{and} \qquad B(s) = \Phi\left(\frac{\partial}{\partial t}\Big|_{(s,0)}\right).$$

Then $(B(0), \dot{A}(0))$ is an element of $(\mathcal{L}^\tau E)_{A(0)}$ and $(A(0), \dot{B}(0))$ is an element of $(\mathcal{L}^\tau E)_{B(0)}$ and the canonical involution maps one into the other. Indeed, let us write $a = A(0), b = B(0), v = \dot{A}(0)$ and $\bar{v} = \dot{B}(0)$. Then, applying the equality $T\chi = \rho \circ \Phi$ to $\frac{\partial}{\partial t}\big|_{(0,0)}$ we get that $\rho(b) = \frac{\partial\chi}{\partial t}(0,0)$ and thus

$$(T_a\tau)(v) = (T_a\tau)(\dot{A}(0)) = \frac{\mathrm{d}(\tau \circ A)}{\mathrm{d}t}\Big|_{t=0} = \frac{\partial\chi}{\partial t}(0,0) = \rho(b),$$

which proves that $(b, v) \in \mathcal{L}^\tau E$ is an element of $(\mathcal{L}^\tau E)_a$. Similarly, applying $T\chi = \rho \circ \Phi$ to $\frac{\partial}{\partial s}\big|_{(0,0)}$ we get that $\rho(a) = \frac{\partial\chi}{\partial s}(0,0)$ and thus

$$(T_b\tau)(\bar{v}) = (T_b\tau)(\dot{B}(0)) = \frac{\mathrm{d}(\tau \circ B)}{\mathrm{d}s}\Big|_{s=0} = \frac{\partial\chi}{\partial s}(0,0) = \rho(a),$$

which proves that $(a, \bar{v}) \in \mathcal{L}^\tau E$ is an element of $(\mathcal{L}^\tau E)_b$. Finally, to prove that $\sigma_E(b, v) = (a, \bar{v})$ we just need to prove that $\bar{v}(\hat{\theta}) = v(\hat{\theta}) + (d^E\theta)(x)(a, b)$, for every $\theta \in \Gamma(E^*)$, with $x = \tau(a) = \tau(b)$. Applying the condition $(\Phi, \chi)^*(d^E\theta) = d^{T\mathbb{R}^2}((\Phi, \chi)^*\theta)$ to $\frac{\partial}{\partial s}\big|_{(0,0)}$ and $\frac{\partial}{\partial t}\big|_{(0,0)}$, we have

$$(d^E\theta)(x)(a, b) = \frac{\partial}{\partial s}\theta(B(s))\Big|_{s=0} - \frac{\partial}{\partial t}\theta(A(t))\Big|_{t=0} = \dot{B}(0)(\hat{\theta}) - \dot{A}(0)(\hat{\theta}) = \bar{v}(\hat{\theta}) - v(\hat{\theta}),$$

which proves $\sigma_E(b, v) = (a, \bar{v})$.

Next, we see that σ_E is a morphism of Lie algebroids where on $pr_1 : \mathcal{L}^\tau E \longrightarrow E$ we have to consider the action Lie algebroid structure that we are going to explain.

It is well known (see, for instance, [12]) that the tangent bundle to E, TE, is a vector bundle over TM with vector bundle projection the tangent map to τ, $T\tau : TE \to TM$. Moreover, if X is a section of $\tau : E \to M$ then the tangent map to X

$$TX : TM \to TE$$

is a section of the vector bundle $T\tau : TE \to TM$. We may also consider the section $\hat{X}^0 : TM \to TE$ of $T\tau : TE \to TM$ given by

$$\hat{X}^0(u) = (T_x 0)(u) + X(x)^v_{0(x)} \tag{4.2}$$

for $u \in T_xM$, where $0 : M \to E$ is the zero section of E and ${}^v_{0(x)} : E_x \to T_{0(x)}(E_x)$ is the canonical isomorphism between E_x and $T_{0(x)}(E_x)$.

If $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ then $\left\{Te_\alpha, \hat{e}^0_\alpha\right\}$ is a local basis of $\Gamma(TE)$. Now, suppose that (x^i) are local coordinates on an open subset U of M and that $\{e_\alpha\}$ is a basis of the vector bundle $\tau^{-1}(U) \to U$. Denote by (x^i, y^α) (respectively, $(x^i, \dot{x}^i)$ and $(x^i, y^\alpha; \dot{x}^i, \dot{y}^\alpha)$) the corresponding coordinates on the open subset $\tau^{-1}(U)$ of E (respectively, $\tau_M^{-1}(U)$ of TM and $\tau_E(\tau^{-1}(U))$ of TE, $\tau_M : TM \to M$ and $\tau_E : TE \to E$ being the canonical projections). Then, the local expression of $T\tau : TE \to TM$ is

$$(T\tau)(x^i, y^\alpha; \dot{x}^i, \dot{y}^\alpha) = (x^i, \dot{x}^i)$$

and the sum and product by real numbers in the vector bundle $T\tau : TE \to TM$ are locally given by

$$(x^i, y^\alpha; \dot{x}^i, \dot{y}^\alpha) \oplus (x^i, \bar{y}^\alpha; \dot{x}^i, \dot{\bar{y}}^\alpha) = (x^i, y^\alpha + \bar{y}^\alpha; \dot{x}^i, \dot{y}^\alpha + \dot{\bar{y}}^\alpha)$$
$$\lambda \cdot (x^i, y^\alpha; \dot{x}^i, \dot{y}^\alpha) = (x^i, \lambda y^\alpha; \dot{x}^i, \lambda \dot{y}^\alpha).$$

Moreover, if X is a section of the vector bundle $\tau : E \to M$ and

$$X = X^\alpha e_\alpha,$$

then

$$\begin{aligned} TX(x^i, \dot{x}^i) &= \left(x^i, X^\alpha; \dot{x}^i, \dot{x}^i \frac{\partial X^\alpha}{\partial x^i}\right), \\ \hat{X}^0(x^i, \dot{x}^i) &= (x^i, 0; \dot{x}^i, X^\alpha). \end{aligned} \tag{4.3}$$

Next, following [26] we define a Lie algebroid structure $([\![\cdot, \cdot]\!]^T, \rho^T)$ on the vector bundle $T\tau : TE \to TM$. The anchor map ρ^T is given by $\rho^T = \sigma_{TM} \circ T(\rho) : TE \to T(TM)$, $\sigma_{TM} : T(TM) \to T(TM)$ being the canonical involution and $T(\rho) : TE \to T(TM)$ the tangent map to $\rho : E \to TM$. The Lie bracket $[\![\cdot, \cdot]\!]^T$ on the space $\Gamma(TE)$ is characterized by the following equalities,

$$\begin{aligned} &[\![TX, TY]\!]^T = T[\![X, Y]\!], \qquad [\![TX, \hat{Y}^0]\!]^T = \widehat{[\![X, Y]\!]}^0 \\ &[\![\hat{X}^0, \hat{Y}^0]\!]^T = 0, \end{aligned} \tag{4.4}$$

for $X, Y \in \Gamma(E)$.

Now, we consider the pull-back vector bundle $\rho^*(TE)$ of $T\tau : TE \to TM$ over the anchor map $\rho : E \to TM$. Note that

$$\rho^*(TE) = \{(b, v) \in E \times TE / \rho(b) = (T\tau)(v)\},$$

that is, the total space of the vector bundle is just $\mathcal{L}^\tau E$, the prolongation of E over τ. Thus, $\rho^*(TE)$ is a vector bundle over E with vector bundle projection $pr_1 : \rho^*(TE) \to E$ given by $pr_1(a, u) = a$, for $(a, u) \in \rho^*(TE)$.

If (x^i) are local coordinates on M, $\{e_\alpha\}$ is a local basis of sections of the vector bundle $\tau : E \to M$ and (x^i, y^α) (respectively, $(x^i, y^\alpha; z^\alpha, v^\alpha)$) are the corresponding coordinates on E (respectively, $\mathcal{L}^\tau E \equiv \rho^*(TE)$) then the local expressions of the vector bundle projection, the sum and the product by real numbers in the vector bundle $\rho^*(TE) \to E$ are

$$pr_1(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, z^\alpha)$$
$$(x^i, y^\alpha; z^\alpha, v^\alpha) \oplus (x^i, \bar{y}^\alpha; z^\alpha, \bar{v}^\alpha) = (x^i, y^\alpha + \bar{y}^\alpha; z^\alpha, v^\alpha + \bar{v}^\alpha)$$
$$\lambda(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, \lambda y^\alpha; z^\alpha, \lambda v^\alpha).$$

Moreover, we have the following result.

Theorem 4.4. *There exists a unique action $\Psi : \Gamma(TE) \to \mathfrak{X}(E)$ of the Lie algebroid $(TE, [\![,]\!]^T, \rho^T)$ over the anchor map $\rho : E \to TM$ such that*

$$\Psi(TX) = X^c, \qquad \Psi(\hat{X}^0) = X^v, \tag{4.5}$$

for $X \in \Gamma(E)$, where X^c (respectively, X^v) is the complete lift (respectively, the vertical lift) of X. In fact, if $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and $\bar{X}$ is a section of $T\tau : TE \to TM$ such that

$$\bar{X} = \bar{f}^\alpha T(e_\alpha) + \bar{g}^\alpha \hat{e}^0_\alpha$$

with $\bar{f}^\alpha, \bar{g}^\alpha$ local real functions on TM, then

$$\Psi(\bar{X}) = (\bar{f}^\alpha \circ \rho) e^c_\alpha + (\bar{g}^\alpha \circ \rho) e^v_\alpha.$$

Proof. A direct computation, using (2.2), (2.25) and (4.3), proves that

$$T\rho \circ X^c = \rho^T \circ TX \circ \rho, \qquad T\rho \circ X^v = \rho^T \circ \hat{X}^0 \circ \rho, \tag{4.6}$$

for $X \in \Gamma(E)$, that is, the vector field X^c (respectively, X^v) is ρ-projectable to the vector field $\rho^T(TX)$ (respectively, $\rho^T(\hat{X}^0)$) on TM.

Now, from (2.22), (4.4) and (4.6), we deduce the result. □

Corollary 4.5. *There exists a Lie algebroid structure $\left([\![\cdot,\cdot]\!]^T_\Psi, \rho^T_\Psi\right)$ on the vector bundle $\rho^*(TE) \to E$ such that if $h^i(\bar{X}_i \circ \rho)$ and $s^j(\bar{Y}_j \circ \rho)$ are two sections of $\rho^*(TE) \to E$ with $h^i, s^j \in C^\infty(E)$ and $\bar{X}_i, \bar{Y}_j$ sections of $T\tau : TE \to TM$ then*

$$\begin{aligned}
&\rho^T_\Psi(h^i(\bar{X}_i \circ \rho)) = h^i \Psi(\bar{X}_i),\\
&[\![h^i(\bar{X}_i \circ \rho), s^j(\bar{Y}_j \circ \rho)]\!]^T_\Psi = h^i s^j([\![\bar{X}_i, \bar{Y}_j]\!]^T \circ \rho) + h^i \Psi(\bar{X}_i)(s^j)(\bar{Y}_j \circ \rho)\\
&\qquad - s^j \Psi(\bar{Y}_j)(h^i)(\bar{X}_i \circ \rho).
\end{aligned}$$

Next, we will give a local description of the Lie algebroid structure $\left([\![\cdot,\cdot]\!]^T_\Psi, \rho^T_\Psi\right)$. For this purpose, we consider local coordinates (x^i) on M and a local basis $\{e_\alpha\}$ of $\Gamma(E)$. Denote by ρ^i_α and $C^\gamma_{\alpha\beta}$ the structure functions of E with respect to (x^i) and $\{e_\alpha\}$, by $(x^i, y^\alpha; z^\alpha, v^\alpha)$ the corresponding coordinates on $\rho^*(TE) \equiv \mathcal{L}^\tau E$ and by $(x^i, y^\alpha; \dot{x}^i, \dot{y}^\alpha)$ the corresponding coordinates on TE.

If X is a section of E and $X = X^\alpha e_\alpha$, we may introduce the sections $T^\rho X$ and $\hat{X}^\rho$ of $\rho^*(TE)$ given by

$$T^\rho X = TX \circ \rho, \qquad \hat{X}^\rho = \hat{X}^0 \circ \rho.$$

We have that (see (4.3))

$$T^\rho X(x^i, z^\alpha) = \left(x^i, X^\alpha; \rho^i_\alpha z^\alpha, \rho^i_\beta z^\beta \frac{\partial X^\alpha}{\partial x^i}\right), \qquad \hat{X}^\rho(x^i, z^\alpha) = (x^i, 0; z^\alpha, X^\alpha). \tag{4.7}$$

Thus, $\left\{T^\rho e_\alpha, \hat{e}^\rho_\alpha\right\}$ is a local basis of $\Gamma(\rho^*(TE))$. Moreover, from (4.4), (4.5) and corollary 4.5, it follows that

$$\begin{aligned}
&[\![T^\rho X, T^\rho Y]\!]^T_\Psi = T^\rho[\![X, Y]\!], \qquad [\![T^\rho X, \hat{Y}^\rho]\!]^T_\Psi = \widehat{[\![X, Y]\!]}^\rho, \qquad [\![\hat{X}^\rho, \hat{Y}^\rho]\!]^T_\Psi = 0,\\
&\rho^T_\Psi(T^\rho X) = X^c, \qquad \rho^T_\Psi(\hat{X}^\rho) = X^v,
\end{aligned} \tag{4.8}$$

for $X, Y \in \Gamma(E)$. In particular,

$$\begin{aligned}
&[\![T^\rho e_\alpha, T^\rho e_\beta]\!]^T_\Psi = T^\rho[\![e_\alpha, e_\beta]\!], \qquad [\![T^\rho e_\alpha, \hat{e}^\rho_\beta]\!]^T_\Psi = \widehat{[\![e_\alpha, e_\beta]\!]}^\rho, \qquad [\![\hat{e}^\rho_\alpha, \hat{e}^\rho_\beta]\!]^T_\Psi = 0,\\
&\rho^T_\Psi(T^\rho e_\alpha) = e^c_\alpha, \qquad \rho^T_\Psi(\hat{e_\alpha}^\rho) = e^v_\alpha.
\end{aligned} \tag{4.9}$$

Therefore,

$$\rho^T_\Psi(x^i, y^\alpha; z^\alpha, v^\alpha) = \left(x^i, z^\alpha; \rho^i_\alpha y^\alpha, v^\gamma + C^\gamma_{\alpha\beta} y^\beta z^\alpha\right), \tag{4.10}$$

$$\begin{aligned}
&[\![T^\rho e_\alpha, T^\rho e_\beta]\!]^T_\Psi(x^i, z^\alpha) = C^\gamma_{\alpha\beta}(T^\rho e_\gamma) + \left(\rho^i_\nu \frac{\partial C^\gamma_{\alpha\beta}}{\partial x^i} z^\nu\right) \hat{e}^\rho_\gamma,\\
&\left[\!\left[T^\rho e_\alpha, \hat{e}^\rho_\beta\right]\!\right]^T_\Psi(x^i, z^\alpha) = C^\gamma_{\alpha\beta} \hat{e}^\rho_\gamma, \qquad \left[\!\left[\hat{e}^\rho_\alpha, \hat{e}^\rho_\beta\right]\!\right]^T_\Psi(x^i, z^\alpha) = 0.
\end{aligned} \tag{4.11}$$

Remark 4.6. If E is the standard Lie algebroid TM then $\rho^*(TE)$ is the tangent bundle to TM and the vector bundle $\rho^*(TE) \to TM$ is $T(TM)$ with vector bundle projection $T\tau_M : T(TM) \to TM$, where $\tau_M : TM \to M$ is the canonical projection. Moreover, following [26], one may consider the tangent Lie algebroid structure $([\cdot,\cdot]^T, \sigma_{TM})$ on the vector bundle $T\tau_M : T(TM) \to TM$ and it is easy to prove that the Lie algebroid structures $([\cdot,\cdot]^T, \sigma_{TM})$ and $\left([\cdot,\cdot]^T_\Psi, (Id)^T_\Psi\right)$ coincide.

Next, we will prove that the canonical involution is a morphism of Lie algebroids.

Theorem 4.7. *The canonical involution σ_E is the unique Lie algebroid morphism $\sigma_E : \mathcal{L}^\tau E \to \rho^*(TE)$ between the Lie algebroids $(\mathcal{L}^\tau E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ and $\left(\rho^*(TE), [\![\cdot,\cdot]\!]^T_\Psi, \rho^T_\psi\right)$ such that σ_E is an involution, that is, $\sigma_E^2 = Id$.*

Proof. Using (2.27), (2.32), (4.1) and (4.7), we deduce that

$$\sigma_E \circ X^{\mathbf{c}} = T^\rho X, \qquad \sigma_E \circ X^{\mathbf{v}} = \hat{X}^\rho, \tag{4.12}$$

for $X \in \Gamma(E)$, where $X^{\mathbf{c}}$ and $X^{\mathbf{v}}$ are the complete and vertical lift, respectively, of X to $\mathcal{L}^\tau E$. Thus, from (2.23), (2.24) and (4.8), we obtain that σ_E is a morphism between the Lie algebroids $(\mathcal{L}^\tau E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ and $\left(\rho^*(TE), [\![\cdot,\cdot]\!]^T_\Psi, \rho^T_\Psi\right)$.

Moreover, if σ'_E is another morphism which satisfies the above conditions then, using that σ'_E is a vector bundle morphism and the fact that σ'_E is an involution, we deduce that the local expression of σ'_E is

$$\sigma'_E(x^i, y^\gamma; z^\gamma, v^\gamma) = (x^i, z^\gamma; y^\gamma, g^\gamma),$$

where g^γ is a linear function in the coordinates (z^γ, v^γ). Now, since

$$\rho^T_\Psi \circ \sigma'_E = \rho^\tau$$

we conclude that (see (2.34) and (4.10))

$$g^\gamma = v^\gamma + C^\gamma_{\alpha\beta} z^\alpha y^\beta, \qquad \text{for all} \quad \gamma.$$

Therefore, $\sigma_E = \sigma'_E$. □

5. Tulczyjew's triple on Lie algebroids

Let $(E, [\![,]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m. Denote by $\tau : E \to M$ the vector bundle projection of E and by $\tau^* : E^* \to M$ the vector bundle projection of the dual bundle E^* to E. Then, TE^* is a vector bundle over TM with vector bundle projection $T\tau^* : TE^* \to TM$ and we may consider the pullback vector bundle $\rho^*(TE^*)$ of $T\tau^* : TE^* \to TM$ over ρ, that is,

$$\rho^*(TE^*) = \{(b, v) \in E \times TE^* / \rho(b) = (T\tau^*)(v)\}.$$

It is clear that $\rho^*(TE^*)$ is the prolongation $\mathcal{L}^{\tau^*}E$ of E over τ^* (see section 3.1). Thus, $\mathcal{L}^{\tau^*}E = \rho^*(TE^*)$ is a vector bundle over E with the vector bundle projection

$$pr_1 : \rho^*(TE^*) \to E, \qquad (b, v) \to pr_1(b, v) = b.$$

Note that if $b_0 \in E$ then

$$(pr_1)^{-1}(b_0) \cong (T\tau^*)^{-1}(\rho(b_0)).$$

Moreover, if (x^i) are coordinates on M, $\{e_\alpha\}$ is a local basis of $\Gamma(E)$, (x^i, y^α) are the corresponding coordinates on E and $(x^i, y_\alpha; z^\alpha, v_\alpha)$ are the corresponding ones on $\rho^*(TE^*) \equiv$

$\mathcal{L}^{\tau^*}E$, then the local expression of the vector bundle projection $pr_1 : \rho^*(TE^*) \equiv \mathcal{L}^{\tau^*}E \to E$, the sum and the product by real numbers in the vector bundle $pr_1 : \rho^*(TE^*) \equiv \mathcal{L}^{\tau^*}E \to E$ are:

$$\begin{aligned}
&pr_1(x^i, y_\alpha; z^\alpha, v_\alpha) = (x^i, z^\alpha),\\
&(x^i, y_\alpha; z^\alpha, v_\alpha) \oplus (x^i, \bar{y}_\alpha; z^\alpha, \bar{v}_\alpha) = (x^i, y_\alpha + \bar{y}_\alpha; z^\alpha, v_\alpha + \bar{v}_\alpha),\\
&\lambda(x^i, y_\alpha; z^\alpha, v_\alpha) = (x^i, \lambda y_\alpha; z^\alpha, \lambda v_\alpha),
\end{aligned}$$

for $(x^i, y_\alpha; z^\alpha, v_\alpha), (x^i, \bar{y}_\alpha; z^\alpha, \bar{v}_\alpha) \in \rho^*(TE^*)$ and $\lambda \in \mathbb{R}$.

Next, we consider the following vector bundles:

- $(\tau^\tau)^* : (\mathcal{L}^\tau E)^* \to E$ (the dual vector bundle to the prolongation of E over τ).
- $pr_1 : \rho^*(TE^*) \to E$ (the pull-back vector bundle of $T\tau^* : TE^* \to TM$ over ρ).
- $\tau^{\tau^*} : \mathcal{L}^{\tau^*}E \to E^*$ (the prolongation of E over $\tau^* : E^* \to M$).
- $(\tau^{\tau^*})^* : (\mathcal{L}^{\tau^*}E)^* \to E^*$ (the dual vector bundle to $\tau^{\tau^*} : \mathcal{L}^{\tau^*}E \to E^*$).

Then, the aim of this section is to introduce two vector bundle isomorphisms

$$A_E : \rho^*(TE^*) \to (\mathcal{L}^\tau E)^*, \qquad \flat_{E^*} : \mathcal{L}^{\tau^*}E \to (\mathcal{L}^{\tau^*}E)^*$$

in such a way that the following diagram is commutative

$$\begin{tikzcd}
(\mathcal{L}^\tau E)^* \arrow[dr, "(\tau^\tau)^*"'] & & \rho^*(TE^*) \equiv \mathcal{L}^{\tau^*}E \arrow[ll, "A_E"'] \arrow[rr, "\flat_{E^*}"] \arrow[dl, "pr_1"] \arrow[dr, "\tau^{\tau^*}"'] & & (\mathcal{L}^{\tau^*}E)^* \arrow[dl, "(\tau^{\tau^*})^*"] \\
& E & & E^* &
\end{tikzcd} \tag{5.1}$$

First, we will define the isomorphism $\flat_{E^*}$.

If Ω_E is the canonical symplectic section of $\mathcal{L}^{\tau^*}E$ then $\flat_{E^*} : \mathcal{L}^{\tau^*}E \to (\mathcal{L}^{\tau^*}E)^*$ is the vector bundle isomorphism induced by Ω_E, that is,

$$\flat_{E^*}(b, v) = i(b, v)(\Omega_E(\tau^{\tau^*}(b, v))),$$

for $(b, v) \in \mathcal{L}^{\tau^*}E$.

If (x^i) are local coordinates on M, $\{e_\alpha\}$ is a local basis of sections of $\Gamma(E)$ and we consider the local coordinates $(x^i, y_\alpha; z^\alpha, v_\alpha)$ on $\mathcal{L}^{\tau^*}E$ (see section 3.1) and the corresponding coordinates on the dual vector bundle $(\mathcal{L}^{\tau^*}E)^*$ then, using (3.7), we deduce that the local expression of $\flat_{E^*}$ in these coordinates is

$$\flat_{E^*}(x^i, y_\alpha; z^\alpha, v_\alpha) = \left(x^i, y_\alpha; -v_\alpha - C^\gamma_{\alpha\beta} y_\gamma z^\beta, z^\alpha\right)$$

where $C^\gamma_{\alpha\beta}$ are the structure functions of the Lie bracket $[\![\cdot, \cdot]\!]$ with respect to the basis $\{e_\alpha\}$.

Now, we define the vector bundle isomorphism

$$A_E : \rho^*(TE^*) \to (\mathcal{L}^\tau E)^*$$

as follows.

Let $\langle \cdot, \cdot \rangle : E \times_M E^* \to \mathbb{R}$ be the natural pairing given by

$$\langle a, a^* \rangle = a^*(a),$$

for $a \in E_x$ and $a^* \in E^*_x$, with $x \in M$. If $b \in E$ and

$$(b, u_a) \in \rho^*(TE)_b, \qquad (b, v_{a^*}) \in \rho^*(TE^*)_b$$

then

$(u_a, v_{a^*}) \in T_{(a,a^*)}(E \times_M E^*) = \{(u'_a, v'_{a^*}) \in T_a E \times T_{a^*} E^* / (T_a \tau)(u'_a) = (T_{a^*} \tau^*)(v'_{a^*})\}$

and we may consider the map

$$\widetilde{T\langle\cdot,\cdot\rangle} : \rho^*(TE) \times_E \rho^*(TE^*) \to \mathbb{R}$$

defined by

$$\widetilde{T\langle\cdot,\cdot\rangle}((b, u_a), (b, v_{a^*})) = \mathrm{d}t_{\langle a,a^*\rangle}((T_{(a,a^*)}\langle\cdot,\cdot\rangle)(u_a, v_{a^*})),$$

where t is the usual coordinate on $\mathbb{R}$ and $T\langle\,,\,\rangle : T(E \times_M E^*) \to T\mathbb{R}$ is the tangent map to $\langle\,,\,\rangle : E \times_M E^* \to \mathbb{R}$.

If $(x^i, y^\alpha; z^\alpha, v^\alpha)$ are local coordinates on $\mathcal{L}^\tau E \equiv \rho^*(TE)$ as in section 2.2.1 then the local expression of $\widetilde{T\langle\cdot,\cdot\rangle}$ is

$$\widetilde{T\langle\cdot,\cdot\rangle}((x^i, y^\alpha; z^\alpha, v^\alpha), (x^i, y_\alpha; z^\alpha, v_\alpha)) = y^\alpha v_\alpha + y_\alpha v^\alpha. \tag{5.2}$$

Thus, the pairing $\widetilde{T\langle\cdot,\cdot\rangle}$ is non-singular and, therefore it induces an isomorphism between the vector bundles $\rho^*(TE) \to E$ and $\rho^*(TE^*)^* \to E$ which we also denote by $\widetilde{T\langle\cdot,\cdot\rangle}$, that is,

$$\widetilde{T\langle\cdot,\cdot\rangle} : \rho^*(TE) \to \rho^*(TE^*)^*.$$

From (5.2), it follows that

$$\widetilde{T\langle\cdot,\cdot\rangle}(x^i, y^\gamma; z^\gamma, v^\gamma) = (x^i, v^\gamma; z^\gamma, y^\gamma). \tag{5.3}$$

Next, we consider the isomorphism of vector bundles $A_E^* : \mathcal{L}^\tau E \to \rho^*(TE^*)^*$ given by

$$A_E^* = \widetilde{T\langle\cdot,\cdot\rangle} \circ \sigma_E$$

$\sigma_E : \mathcal{L}^\tau E \to \rho^*(TE)$ being the canonical involution introduced in section 4. Using (5.3) and theorem 4.7, we deduce that the local expression of the map A_E^* is

$$A_E^*(x^i, y^\gamma; z^\gamma, v^\gamma) = \left(x^i, v^\gamma + C^\gamma_{\alpha\beta} y^\beta z^\alpha; y^\gamma, z^\gamma\right). \tag{5.4}$$

Finally, the isomorphism $A_E : \rho^*(TE^*) \to (\mathcal{L}^\tau E)^*$ between the vector bundles $\rho^*(TE^*) \to E$ and $(\mathcal{L}^\tau E)^* \to E$ is just the dual map to $A_E^* : \mathcal{L}^\tau E \to \rho^*(TE^*)^*$. From (5.4) we conclude that

$$A_E(x^i, y_\alpha; z^\alpha, v_\alpha) = \left(x^i, z^\alpha; v_\alpha + C^\gamma_{\alpha\beta} y_\gamma z^\beta, y_\alpha\right). \tag{5.5}$$

Diagram (5.1) will be called *Tulczyjew's triple for the Lie algebroid E.*

Remark 5.1. If E is the standard Lie algebroid TM then the vector bundle isomorphisms $A_{TM} : T(T^*M) \to T^*(TM)$ and $\flat_{T^*M} : T(T^*M) \to T^*(T^*M)$ were considered by Tulczyjew [45, 46] and diagram (5.1) is just Tulczyjew's triple.

6. The prolongation of a symplectic Lie algebroid

First of all, we will introduce the definition of a symplectic Lie algebroid.

Definition 6.1. *A Lie algebroid $(E, [\![\cdot,\cdot]\!], \rho)$ over a manifold M is said to be symplectic if it admits a symplectic section Ω, that is, Ω is a section of the vector bundle $\wedge^2 E^* \to M$ such that:*

(i) *For all* $x \in M$, *the 2-form* $\Omega(x) : E_x \times E_x \to \mathbb{R}$ *on the vector space* E_x *is nondegenerate and*
(ii) Ω *is a 2-cocycle, i.e.,* $d^E\Omega = 0$.

Example 6.2.

(i) Let (M, Ω) be a symplectic manifold. Then the tangent bundle TM to M is a symplectic Lie algebroid.
(ii) Let $(E, [\![\cdot,\cdot]\!], \rho)$ be an arbitrary Lie algebroid and $([\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ be the Lie algebroid structure on the prolongation $\mathcal{L}^{\tau^*}E$ of E over the bundle projection $\tau^* : E^* \to M$ of the dual vector bundle E^* to E. Then $(\mathcal{L}^{\tau^*}E, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ is a symplectic Lie algebroid and Ω_E is a symplectic section of $\mathcal{L}^{\tau^*}E$, Ω_E being the canonical symplectic 2-section of $\mathcal{L}^{\tau^*}E$ (see section 3.2).

In this section, we will prove that the prolongation of a symplectic Lie algebroid over the vector bundle projection is a symplectic Lie algebroid.

For this purpose, we will need some previous results.

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $\tau : E \to M$ be the vector bundle projection. Denote by $([\![\cdot,\cdot]\!]^{\tau}, \rho^{\tau})$ the Lie algebroid structure on the prolongation $\mathcal{L}^{\tau}E$ of E over τ. If $pr_1 : E \times TE \to E$ is the canonical projection on the first factor then the pair $(pr_{1|\mathcal{L}^{\tau}E}, \tau)$ is a morphism between the Lie algebroids $(\mathcal{L}^{\tau}E, [\![\cdot,\cdot]\!]^{\tau}, \rho^{\tau})$ and $(E, [\![\cdot,\cdot]\!], \rho)$ (see section 2.1.1). Thus, if $\alpha \in \Gamma(\wedge^k E^*)$ we may consider the section $\alpha^{\mathbf{v}}$ of the vector bundle $\wedge^k(\mathcal{L}^{\tau}E)^* \to E$ defined by

$$\alpha^{\mathbf{v}} = (pr_{1|\mathcal{L}^{\tau}E}, \tau)^*(\alpha). \tag{6.1}$$

$\alpha^{\mathbf{v}}$ is called *the vertical lift* to $\mathcal{L}^{\tau}E$ of α and we have that

$$d^{\mathcal{L}^{\tau}E}\alpha^{\mathbf{v}} = (d^E\alpha)^{\mathbf{v}}. \tag{6.2}$$

Note that if $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and $\{e^\alpha\}$ is the dual basis to $\{e_\alpha\}$ then

$$(e^\alpha)^{\mathbf{v}}((e_\beta)^{\mathbf{c}}) = \delta_{\alpha\beta}, \qquad (e^\alpha)^{\mathbf{v}}((e_\beta)^{\mathbf{v}}) = 0,$$

for all α and β. Moreover, if $\gamma \in \Gamma(\wedge^k E^*)$ and

$$\gamma = \gamma_{\alpha_1,\ldots,\alpha_k} e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_k}$$

we have that

$$\gamma^{\mathbf{v}} = \big(\gamma_{\alpha_1,\ldots,\alpha_k} \circ \tau\big)(e^{\alpha_1})^{\mathbf{v}} \wedge \cdots \wedge (e^{\alpha_k})^{\mathbf{v}}.$$

Now, suppose that $f \in C^\infty(M)$. Then, one may consider the complete lift f^c of f to E. f^c is a real C^∞-function on E (see (2.21)). This construction may be generalized as follows.

Proposition 6.3. *If* α *is a section of the vector bundle* $\wedge^k E^* \to M$, *then there exists a unique section* $\alpha^{\mathbf{c}}$ *of the vector bundle* $\wedge^k(\mathcal{L}^{\tau}E)^* \to E$ *such that*

$$\begin{aligned} &\alpha^{\mathbf{c}}\big(X_1^{\mathbf{c}}, \ldots, X_k^{\mathbf{c}}\big) = \alpha(X_1, \ldots, X_k)^c, \\ &\alpha^{\mathbf{c}}\big(X_1^{\mathbf{v}}, X_2^{\mathbf{c}}, \ldots, X_k^{\mathbf{c}}\big) = \alpha(X_1, \ldots, X_k)^v, \\ &\alpha^{\mathbf{c}}\big(X_1^{\mathbf{v}}, \ldots, X_s^{\mathbf{v}}, X_{s+1}^{\mathbf{c}}, \ldots, X_k^{\mathbf{c}}\big) = 0, \qquad \text{if} \quad 2 \leqslant s \leqslant k, \end{aligned} \tag{6.3}$$

for $X_1, \ldots, X_k \in \Gamma(E)$. *Furthermore,*

$$d^{\mathcal{L}^{\tau}E}\alpha^{\mathbf{c}} = (d^E\alpha)^{\mathbf{c}}. \tag{6.4}$$

Proof. We recall that if $\{X_i\}$ is a local basis of $\Gamma(E)$ then $\{X_i^{\mathbf{c}}, X_i^{\mathbf{v}}\}$ is a local basis of $\Gamma(\mathcal{L}^{\tau}E)$.

On the other hand, if $X \in \Gamma(E)$ and $f, g \in C^\infty(M)$ then

$$(fX)^{\mathbf{c}} = f^c X^{\mathbf{v}} + f^v X^{\mathbf{c}}, \qquad (fX)^{\mathbf{v}} = f^v X^{\mathbf{v}},$$
$$(fg)^c = f^c g^v + f^v g^c, \qquad (fg)^v = f^v g^v.$$

Using the above facts, we deduce the first part of the proposition.

Now, if $\alpha \in \Gamma(E^*)$ then, from (2.24) and (6.3), it follows that

$$(d^{\mathcal{L}^\tau E}\alpha^{\mathbf{c}})(X^{\mathbf{c}}, Y^{\mathbf{c}}) = ((d^E\alpha)(X,Y))^c = (d^E\alpha)^{\mathbf{c}}(X^{\mathbf{c}}, Y^{\mathbf{c}})$$
$$(d^{\mathcal{L}^\tau E}\alpha^{\mathbf{c}})(X^{\mathbf{v}}, Y^{\mathbf{c}}) = ((d^E\alpha)(X,Y))^v = (d^E\alpha)^{\mathbf{c}}(X^{\mathbf{v}}, Y^{\mathbf{c}})$$
$$(d^{\mathcal{L}^\tau E}\alpha^{\mathbf{c}})(X^{\mathbf{v}}, Y^{\mathbf{v}}) = 0 = (d^E\alpha)^{\mathbf{c}}(X^{\mathbf{v}}, Y^{\mathbf{v}})$$

for $X, Y \in \Gamma(E)$. Thus,

$$d^{\mathcal{L}^\tau E}\alpha^{\mathbf{c}} = (d^E\alpha)^{\mathbf{c}}, \qquad \text{for} \quad \alpha \in \Gamma(E^*). \tag{6.5}$$

In addition, using (2.24) and (6.3), we have that

$$d^{\mathcal{L}^\tau E} f^c = (d^E f)^c, \qquad \text{for} \quad f \in C^\infty(M) = \Gamma(\Lambda^0 E^*). \tag{6.6}$$

Moreover, using (6.1) and (6.3) we obtain that

$$(f\alpha_1 \wedge \cdots \wedge \alpha_r)^{\mathbf{c}} = f^c \alpha_1^{\mathbf{v}} \wedge \cdots \wedge \alpha_r^{\mathbf{v}} + f^v \sum_{i=1}^r \alpha_1^{\mathbf{v}} \wedge \cdots \wedge \alpha_i^{\mathbf{c}} \wedge \cdots \wedge \alpha_r^{\mathbf{v}} \tag{6.7}$$

for $\alpha_i \in \Gamma(\wedge^{k_i} E^*)$, $i \in \{1, \ldots, r\}$. Therefore, from (6.2), (6.5), (6.6) and (6.7) we conclude that (6.4) holds. □

The section $\alpha^{\mathbf{c}}$ of the vector bundle $\wedge^k((\mathcal{L}^\tau E)^*) \to E$ is called *the complete lift of* α.

If $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and $\{e^\alpha\}$ is the dual basis to $\{e_\alpha\}$ then

$$(e^\alpha)^{\mathbf{c}}((e_\beta)^{\mathbf{c}}) = 0, \qquad (e^\alpha)^{\mathbf{c}}((e_\beta)^{\mathbf{v}}) = \delta_{\alpha\beta},$$

for all α and β. Furthermore, if $\gamma \in \Gamma(\wedge^k E^*)$ and $\gamma = \gamma_{\alpha_1\cdots\alpha_k} e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_k}$, we have that

$$\gamma^{\mathbf{c}} = \gamma^c_{\alpha_1\cdots\alpha_k}(e^{\alpha_1})^{\mathbf{v}} \wedge \cdots \wedge (e^{\alpha_k})^{\mathbf{v}} + \sum_{i=1}^k \big(\gamma_{\alpha_1\cdots\alpha_k} \circ \tau\big)(e^{\alpha_1})^{\mathbf{v}} \wedge \cdots \wedge (e^{\alpha_i})^{\mathbf{c}} \wedge \cdots \wedge (e^{\alpha_k})^{\mathbf{v}}.$$

Thus, if (x^i) are local coordinates defined on an open subset U of M and $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ on U then

$$\begin{aligned}\gamma^{\mathbf{c}}(x^i, y^\alpha) = {} & \rho^i_\beta \frac{\partial \gamma_{\alpha_1\cdots\alpha_k}}{\partial x^i} y^\beta (e^{\alpha_1})^{\mathbf{v}} \wedge \cdots \wedge (e^{\alpha_k})^{\mathbf{v}} \\ & + \sum_j \big(\gamma_{\alpha_1\cdots\alpha_k} \circ \tau\big)(e^{\alpha_1})^{\mathbf{v}} \wedge \cdots \wedge (e^{\alpha_j})^{\mathbf{c}} \wedge \cdots \wedge (e^{\alpha_k})^{\mathbf{v}},\end{aligned}$$

where (x^i, y^α) are the corresponding local coordinates on E and ρ^i_β are the components of the anchor map ρ with respect to (x^i) and to $\{e_\alpha\}$.

Note that

$$\{(e^\alpha)^{\mathbf{c}}, (e^\alpha)^{\mathbf{v}}\}$$

is a local basis of $\Gamma((\mathcal{L}^\tau E)^*)$. In fact, $\{(e^\alpha)^{\mathbf{c}}, (e^\alpha)^{\mathbf{v}}\}$ is the dual basis to the local basis of $\Gamma(\mathcal{L}^\tau E)$

$$\{(e_\alpha)^{\mathbf{v}}, (e_\alpha)^{\mathbf{c}}\}.$$

Remark 6.4. If E is the standard Lie algebroid $\tau_M : TM \to M$ and α is a k-form on M, that is, $\alpha \in \Gamma(\wedge^k(T^*M))$ then $\mathcal{L}^{\tau_M}E = T(TM)$ and $\alpha^{\mathbf{c}}$ is the usual complete lift of α to TM (see [47]).

Next, we will prove the main result of this section.

Theorem 6.5. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a symplectic Lie algebroid with symplectic section Ω and $\tau : E \to M$ be the vector bundle projection of E. Then, the prolongation $\mathcal{L}^\tau E$ of E over τ is a symplectic Lie algebroid and the complete lift $\Omega^{\mathbf{c}}$ of Ω to $\mathcal{L}^\tau E$ is a symplectic section of $\mathcal{L}^\tau E$.*

Proof. It is clear that

$$d^{\mathcal{L}^\tau E}\Omega^{\mathbf{c}} = (d^E\Omega)^{\mathbf{c}} = 0.$$

Moreover, if $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ on an open subset U of M and $\{e^\alpha\}$ is the dual basis to $\{e_\alpha\}$ then

$$\Omega = \tfrac{1}{2}\Omega_{\alpha\beta}e^\alpha \wedge e^\beta$$

with $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$ real functions on U.

Thus, using (6.7), we have that

$$\Omega^{\mathbf{c}} = \tfrac{1}{2}\Omega^c_{\alpha\beta}(e^\alpha)^{\mathbf{v}} \wedge (e^\beta)^{\mathbf{v}} + \Omega^v_{\alpha\beta}(e^\alpha)^{\mathbf{c}} \wedge (e^\beta)^{\mathbf{v}}.$$

Therefore, the local matrix associated with $\Omega^{\mathbf{c}}$ with respect to the basis $\{(e^\alpha)^{\mathbf{c}}, (e^\alpha)^{\mathbf{v}}\}$ is

$$\begin{pmatrix} 0 & (\Omega_{\alpha\beta})^v \\ -(\Omega_{\alpha\beta})^v & (\Omega_{\alpha\beta})^c \end{pmatrix}$$

Consequently, the rank of $\Omega^{\mathbf{c}}$ is $2n$ and $\Omega^{\mathbf{c}}$ is nondegenerate. □

Remark 6.6. Let (M, Ω) be a symplectic manifold. Then, using theorem 6.5, we deduce a well-known result (see [46]): the tangent bundle to M is a symplectic manifold and the complete lift Ω^c of Ω to TM is a symplectic 2-form on TM.

Example 6.7. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $\tau : E \to M$ be the vector bundle projection. Denote by λ_E and Ω_E the Liouville section and the canonical symplectic section of the prolongation $\mathcal{L}^{\tau^*}E$ of E over the vector bundle projection $\tau^* : E^* \to M$ of the dual vector bundle E^* to E.

On the other hand, if (x^i) are coordinates on M and $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ then we may consider the corresponding coordinates (x^i, y_α) (respectively, $(x^i, y_\alpha; z^\alpha, v_\alpha)$) of E^* (respectively, $\mathcal{L}^{\tau^*}E$) and the corresponding local basis $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ of $\Gamma(\mathcal{L}^{\tau^*}E)$ (see section 3.1). Thus, if $\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*}E)$ is the prolongation of $\mathcal{L}^{\tau^*}E$ over the vector bundle projection $\tau^{\tau^*} : \mathcal{L}^{\tau^*}E \to E^*$ then

$$\left\{\tilde{e}^{\mathbf{c}}_\alpha, \bar{e}^{\mathbf{c}}_\alpha, \tilde{e}^{\mathbf{v}}_\alpha, \bar{e}^{\mathbf{v}}_\alpha\right\}$$

is a local basis of $\Gamma(\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*}E))$ and if $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ is the dual basis to $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ then

$$\{(\tilde{e}^\alpha)^{\mathbf{v}}, (\bar{e}^\alpha)^{\mathbf{v}}, (\tilde{e}^\alpha)^{\mathbf{c}}, (\bar{e}^\alpha)^{\mathbf{c}}\}$$

is the dual basis of $\left\{\tilde{e}^{\mathbf{c}}_\alpha, \bar{e}^{\mathbf{c}}_\alpha, \tilde{e}^{\mathbf{v}}_\alpha, \bar{e}^{\mathbf{v}}_\alpha\right\}$.

Moreover, using (3.7) and (6.7), we deduce that the local expressions of the complete lifts $\lambda^{\mathbf{c}}_E$ and $\Omega^{\mathbf{c}}_E$ of λ_E and Ω_E are

$$\begin{aligned}\lambda^{\mathbf{c}}_E(x^i, y_\alpha; z^\alpha, v_\alpha) &= y^c_\alpha(\tilde{e}^\alpha)^{\mathbf{v}} + y^v_\alpha(\tilde{e}^\alpha)^{\mathbf{c}},\\ \Omega^{\mathbf{c}}_E(x^i, y_\alpha; z^\alpha, v_\alpha) &= (\tilde{e}^\alpha)^{\mathbf{c}} \wedge (\bar{e}^\alpha)^{\mathbf{v}} + (\tilde{e}^\alpha)^{\mathbf{v}} \wedge (\bar{e}^\alpha)^{\mathbf{c}} + \tfrac{1}{2}\left(C^\gamma_{\alpha\beta}y_\gamma\right)^c(\tilde{e}^\alpha)^{\mathbf{v}} \wedge (\tilde{e}^\beta)^{\mathbf{v}}\\ &\quad + C^\gamma_{\alpha\beta}y_\gamma(\tilde{e}^\alpha)^{\mathbf{c}} \wedge (\tilde{e}^\beta)^{\mathbf{v}}.\end{aligned}$$

Therefore, from (3.2), we conclude that

$$\lambda_E^{\mathbf{c}}(x^i, y_\alpha; z^\alpha, v_\alpha) = y_\alpha(\tilde{e}^\alpha)^{\mathbf{c}} + v_\alpha(\tilde{e}^\alpha)^{\mathbf{v}},$$

$$\Omega_E^{\mathbf{c}}(x^i, y_\alpha; z^\alpha, v_\alpha) = (\tilde{e}^\alpha)^{\mathbf{c}} \wedge (\bar{e}^\alpha)^{\mathbf{v}} + (\tilde{e}^\alpha)^{\mathbf{v}} \wedge (\bar{e}^\alpha)^{\mathbf{c}} + \frac{1}{2}\left(\rho_\mu^i \frac{\partial C_{\alpha\beta}^\gamma}{\partial x^i} z^\mu y_\gamma + C_{\alpha\beta}^\gamma v_\gamma\right)(\tilde{e}^\alpha)^{\mathbf{v}} \wedge (\tilde{e}^\beta)^{\mathbf{v}}$$
$$+ C_{\alpha\beta}^\gamma y_\gamma (\tilde{e}^\alpha)^{\mathbf{c}} \wedge (\tilde{e}^\beta)^{\mathbf{v}}, \tag{6.8}$$

ρ_α^i and $C_{\alpha\beta}^\gamma$ being the structure functions of the Lie algebroid $(E, [\![\cdot,\cdot]\!], \rho)$ with respect to the coordinates (x^i) and to the basis $\{e_\alpha\}$.

7. Lagrangian Lie subalgebroids in symplectic Lie algebroids

First of all, we will introduce the notion of a Lagrangian Lie subalgebroid of a symplectic Lie algebroid.

Definition 7.1. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a symplectic Lie algebroid with symplectic section Ω and $j : F \to E$, $i : N \to M$ be a Lie subalgebroid (see remark 2.2). Then, the Lie subalgebroid is said to be Lagrangian if $j(F_x)$ is a Lagrangian subspace of the symplectic vector space $(E_{i(x)}, \Omega_{i(x)})$, for all $x \in N$.*

Definition 7.1 implies that:

(i) rank $F = \frac{1}{2}$rank E and
(ii) $(\Omega(i(x)))_{|j(F_x)\times j(F_x)} = 0$, for all $x \in N$.

Remark 7.2. Let (M, Ω) be a symplectic manifold, S be a submanifold of M and $i : S \to M$ be the canonical inclusion. Then, the standard Lie algebroid $\tau_M : TM \to M$ is symplectic and $i : S \to M$, $j = Ti : TS \to TM$ is a Lie subalgebroid of $\tau_M : TM \to M$. Moreover, one may prove that S is a Lagrangian submanifold of M in the usual sense if and only if the Lie subalgebroid $i : S \to M$, $j = Ti : TS \to TM$ of $\tau_M : TM \to M$ is Lagrangian.

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over M. Then, the prolongation $\mathcal{L}^{\tau^*}E$ of E over the vector bundle projection $\tau^* : E^* \to M$ is a symplectic Lie algebroid. Moreover, if x is a point of M and E_x^* is the fibre of E^* over the point x, we will denote by

$$j_x : TE_x^* \to \mathcal{L}^{\tau^*}E, \qquad i_x : E_x^* \to E^*$$

the maps given by

$$j_x(v) = (0(x), v), \qquad i_x(\alpha) = \alpha$$

for $v \in TE_x^*$ and $\alpha \in E_x^*$, where $0 : M \to E$ is the zero section of E. Note that if $v \in TE_x^*$, $(T\tau^*)(v) = 0$ and thus $(0(x), v) \in \mathcal{L}^{\tau^*}E$.

On the other hand, if $\gamma \in \Gamma(E^*)$ we will denote by F_γ the vector bundle over $\gamma(M)$ given by

$$F_\gamma = \{(b, (T\gamma)(\rho(b))) \in E \times TE^*/b \in E\} \tag{7.1}$$

and by $j_\gamma : F_\gamma \to \mathcal{L}^{\tau^*}E$ and $i_\gamma : \gamma(M) \to E^*$ the canonical inclusions. Note that the pair $((Id, T\gamma \circ \rho), \gamma)$ is an isomorphism between the vector bundles E and F_γ, where the map $(Id, T\gamma \circ \rho) : E \to F_\gamma$ is given by

$$(Id, T\gamma \circ \rho)(b) = (b, (T\gamma)(\rho(b))), \qquad \text{for} \quad b \in E.$$

Thus, F_γ is a Lie algebroid over $\gamma(M)$. Moreover, we may prove the following results.

Proposition 7.3. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $\mathcal{L}^{\tau^*}E$ be the prolongation of E over the vector bundle projection $\tau^* : E^* \to M$.*

(i) *If x is a point of M then $j_x : TE_x^* \to \mathcal{L}^{\tau^*}E$ and $i_x : E_x^* \to E^*$ is a Lagrangian Lie subalgebroid of the symplectic Lie algebroid $\mathcal{L}^{\tau^*}E$.*

(ii) *If $\gamma \in \Gamma(E^*)$ then $j_\gamma : F_\gamma \to \mathcal{L}^{\tau^*}E$ and $i_\gamma : \gamma(M) \to E^*$ is a Lagrangian Lie subalgebroid of the symplectic Lie algebroid $\mathcal{L}^{\tau^*}E$ if and only if γ is a 1-cocycle for the cohomology complex of the Lie algebroid E, that is, $d^E\gamma = 0$.*

Proof. (i) It is clear that the rank of the vector bundle $\tau_{E_x^*} : TE_x^* \to E_x^*$ is $n = \frac{1}{2}\mathrm{rank}\,(\mathcal{L}^{\tau^*}E)$. Moreover, if (x^i) are local coordinates on M, $\{e_\alpha\}$ is a local basis of $\Gamma(E)$, (x^i, y_α) are the corresponding coordinates on E^* and $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ is the corresponding local basis of $\Gamma(\mathcal{L}^{\tau^*}E)$ then, from (3.1), it follows that

$$(j_x, i_x)^*(\tilde{e}^\alpha) = 0, \qquad (j_x, i_x)^*(\bar{e}^\alpha) = d^{TE_x^*}(y_\alpha \circ i_x),$$
$$j_x\left(\frac{\partial}{\partial y_\alpha}_{|\mu}\right) = \bar{e}_\alpha(\mu), \qquad \text{for all} \quad \mu \in E_x^*. \tag{7.2}$$

This, using (3.3), implies that $j_x : TE_x^* \to \mathcal{L}^{\tau^*}E$ and $i_x : E_x^* \to E^*$ is a morphism of Lie algebroids. Thus, since j_x is injective and i_x is an injective immersion, we deduce that $j_x : TE_x^* \to \mathcal{L}^{\tau^*}E$ and $i_x : E_x^* \to E^*$ is a Lie subalgebroid of $\mathcal{L}^{\tau^*}E$. Finally, from (3.7) and (7.2), we conclude that

$$(\Omega_E(i_x(\mu)))_{|j_x(T_\mu E_x^*)\times j_x(T_\mu E_x^*)} = 0,$$

for all $\mu \in E_x^*$.

(ii) If γ is a section of E^* then the Lie algebroids $E \to M$ and $F_\gamma \to \gamma(M)$ are isomorphic and, under this isomorphism, the inclusions j_γ and i_γ are the maps $(Id, T\gamma \circ \rho) : E \to \mathcal{L}^{\tau^*}E$ and $\gamma : M \to E^*$, respectively. Furthermore, it clear that the map $(Id, T\gamma \circ \rho)$ is injective and that $\gamma : M \to E^*$ is an injective immersion.

On the other hand, from theorem 3.4, we have that

$$((Id, T\gamma \circ \rho), \gamma)^*(\Omega_E) = -d\gamma.$$

Therefore, the Lie subalgebroid $j_\gamma : F_\gamma \to \mathcal{L}^{\tau^*}E$ and $i_\gamma : \gamma(M) \to E^*$ is Lagrangian if and only if γ is a 1-cocycle. □

Remark 7.4. Using remark 7.2 and applying proposition 7.3 to the particular case when E is the standard Lie algebroid TM, we deduce two well-known results (see, for instance, [1]):

(i) If Ω_{TM} is the canonical symplectic 2-form on T^*M and x is a point of M then the cotangent space to M at x, T_x^*M, is a Lagrangian submanifold of the symplectic manifold (T^*M, Ω_{TM}).

(ii) If $\gamma : M \to T^*M$ is a 1-form on M then the submanifold $\gamma(M)$ is Lagrangian in the symplectic manifold (T^*M, Ω_M) if and only if γ is a closed 1-form.

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and S be a submanifold of E. Denote by $\tau : E \to M$ the vector bundle projection, by $i : S \to E$ the canonical inclusion and by $\tau^S : S \to M$ the map given by

$$\tau^S = \tau \circ i.$$

If there exists a natural number c such that

$$\dim(\rho(E_{\tau^S(x)}) + (T_x\tau^S)(T_xS)) = c, \qquad \text{for all} \quad x \in S, \tag{7.3}$$

then, using the results of section 2.1.1, one can consider the prolongation of the Lie algebroid E over the map τ^S,

$$\mathcal{L}^{\tau^S}E = \{(b, v) \in E \times TS / \rho(b) = (T\tau^S)(v)\},$$

which is a Lie algebroid over S of rank $n+s-c$, where $s = \dim S$. Moreover, the maps $(Id, Ti) : \mathcal{L}^{\tau^S}E \to \mathcal{L}^{\tau}E$ and $i : S \to E$ define a Lie subalgebroid of the prolongation of E over the bundle projection τ, where (Id, Ti) is the map given by

$$(Id, Ti)(b, v) = (b, Ti(v)), \qquad \text{for} \quad (b, v) \in \mathcal{L}^{\tau^S}E.$$

Examples 7.5. (i) Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold M and X be a section of E. Suppose that the submanifold S is $X(M)$. Then, it is easy to prove that condition (7.3) holds and that c is just $m = \dim M = \dim S$. Thus, one may consider the prolongation $\mathcal{L}^{\tau^S}E$ of E over the map $\tau^S : S = X(M) \to M$ and the rank of the vector bundle $\mathcal{L}^{\tau^S}E \to S$ is n.

Note that if x is a point of M, the fibre of $\mathcal{L}^{\tau^S}E$ over $X(x) \in S$ is the vector space

$$(\mathcal{L}^{\tau^S}E)_{X(x)} = \{(b, (T_xX)(\rho(b)))/b \in E_x\}.$$

Therefore, if Y is a section of E then, using (2.25), we deduce that

$$TX(\rho(Y)) = (Y^c - [\![X, Y]\!]^v) \circ X,$$

where Z^c (respectively, Z^v) denotes the complete (respectively, vertical) lift of a section Z of E to a section of the vector bundle $TE \to E$. Consequently (see (2.23)),

$$TX(\rho(Y)) = \{\rho^{\tau}(Y^{\mathbf{c}} - [\![X, Y]\!]^{\mathbf{v}})\} \circ X. \tag{7.4}$$

Here, $Z^{\mathbf{c}}$ (respectively, $Z^{\mathbf{v}}$) is the complete (respectively, vertical) lift of a section Z of E to a section of the vector bundle $\mathcal{L}^{\tau}E \to E$. Now, from (7.4), we obtain that

$$Y^{\mathbf{c}}_{|S} - [\![X, Y]\!]^{\mathbf{v}}_{|S}$$

is a section of the vector bundle $\mathcal{L}^{\tau^s}E \to S$. Thus, if $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ it follows that

$$\left\{e^{\mathbf{c}}_{\alpha|S} - [\![X, e_\alpha]\!]^{\mathbf{v}}_{|S}\right\}$$

is a local basis of $\Gamma(\mathcal{L}^{\tau^S}E)$.

(ii) Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold of M of dimension m. Denote by $\mathcal{L}^{\tau^*}E$ the prolongation of E over the vector bundle projection $\tau^* : E^* \to M$. $\mathcal{L}^{\tau^*}E$ is a Lie algebroid over E^*. On the other hand, let $\rho^*(TE^*)$ be the pull-back of the vector bundle $T\tau^* : TE^* \to TM$ over the anchor map $\rho : E \to TM$. $\rho^*(TE^*)$ is a vector bundle over E. Moreover, as we know, the total spaces of these vector bundles coincide, that is,

$$\rho^*(TE^*) = \mathcal{L}^{\tau^*}E.$$

Now, suppose that $\tilde{X}$ is a section of the vector bundle $\rho^*(TE^*) \to E$. Then, $S = \tilde{X}(E)$ is a submanifold of $\mathcal{L}^{\tau^*}E$ of dimension $m+n$. Furthermore, if we consider on $\mathcal{L}^{\tau^*}E$ the Lie algebroid structure $([\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ (see section 3.1) then condition (7.3) holds for the submanifold S and the natural number c is $n+m = \dim S$. Thus, the prolongation $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E)$ of the Lie algebroid $(\mathcal{L}^{\tau^*}E, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ over the map $\tau^{\tau^*} \circ i = (\tau^{\tau^*})^S : S \to \mathcal{L}^{\tau^*}E \to E^*$ is a Lie algebroid over S of rank $2n$. In fact, if (x^i) are local coordinates on an open subset U of M, $\{e_\alpha\}$ is a basis of $\tau^{-1}(U) \to U$, $(x, y) \equiv (x^i, y^\alpha)$ (respectively $(x, y; z, v) \equiv (x^i, y_\alpha; z^\alpha, v_\alpha)$) are the corresponding coordinates on E (respectively, $\mathcal{L}^{\tau^*}E$) and the local expression of $\tilde{X}$ in these coordinates is

$$\tilde{X}(x, y) = \tilde{X}(x^i, y^\alpha) = (x^i, \tilde{X}^\alpha; y^\alpha, \tilde{X}'^\alpha)$$

then $\left\{\tilde{e}^{\tilde{X}}_\alpha, \bar{e}^{\tilde{X}}_\alpha\right\}$ is a local basis of sections of $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E) \to S$, where

$$\tilde{e}^{\tilde{X}}_\alpha : S \to \mathcal{L}^{\tau^*}E \times TS, \qquad \bar{e}^{\tilde{X}}_\alpha : S \to \mathcal{L}^{\tau^*}E \times TS$$

are defined by

$$\tilde{e}_\alpha^{\tilde{X}}(\tilde{X}(x,y)) = \left(\tilde{e}_\alpha((\tau^{\tau^*})^S(\tilde{X}(x,y))) + \rho_\alpha^i(x)\frac{\partial\tilde{X}^\beta}{\partial x^i}_{|(x,y)}\bar{e}_\beta((\tau^{\tau^*})^S(\tilde{X}(x,y))),\right.$$
$$\left.\rho_\alpha^i(x)\frac{\partial}{\partial x^i}_{|\tilde{X}(x,y)} + \rho_\alpha^i(x)\frac{\partial\tilde{X}^\beta}{\partial x^i}_{|(x,y)}\frac{\partial}{\partial y_\beta}_{|\tilde{X}(x,y)} + \rho_\alpha^i(x)\frac{\partial\tilde{X}'^\beta}{\partial x^i}_{|(x,y)}\frac{\partial}{\partial v_\beta}_{|\tilde{X}(x,y)}\right),$$
$$\bar{e}_\alpha^{\tilde{X}}(\tilde{X}(x,y)) = \left(\frac{\partial\tilde{X}^\beta}{\partial y^\alpha}_{|(x,y)}\bar{e}_\beta((\tau^{\tau^*})^S(\tilde{X}(x,y))), \frac{\partial}{\partial z^\alpha}_{|\tilde{X}(x,y)}\right.$$
$$\left.+\frac{\partial\tilde{X}^\beta}{\partial y^\alpha}_{|(x,y)}\frac{\partial}{\partial y_\beta}_{|\tilde{X}(x,y)} + \frac{\partial\tilde{X}'^\beta}{\partial y^\alpha}_{|(x,y)}\frac{\partial}{\partial v_\beta}_{|\tilde{X}(x,y)}\right). \tag{7.5}$$

Here, ρ_α^i are the components of the anchor map with respect to (x^i) and $\{e_\alpha\}$ and $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ is the corresponding local basis of $\Gamma(\mathcal{L}^{\tau^*}E)$. Using the local basis $\{\tilde{e}_\alpha^{\tilde{X}}, \bar{e}_\alpha^{\tilde{X}}\}$ of $\Gamma(\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E))$ one may introduce, in a natural way, local coordinates $(x^i, y^\alpha; z_{\tilde{X}}^\alpha, v_{\tilde{X}}^\alpha)$ on $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E)$ as follows. If $\omega_{\tilde{X}} \in \mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E)_{\tilde{X}(a)}$, with $a \in E$, then (x^i, y^α) are the coordinates of a and

$$\omega_{\tilde{X}} = z_{\tilde{X}}^\alpha\tilde{e}_\alpha^{\tilde{X}}(\tilde{X}(a)) + v_{\tilde{X}}^\alpha\bar{e}_\alpha^{\tilde{X}}(\tilde{X}(a)).$$

Moreover, if $([\![\cdot,\cdot]\!]^S, \rho^S)$ is the Lie algebroid structure on the vector bundle $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E) \to S$ then, using (2.13), (2.14), (3.2) and (7.5), we obtain that

$$[\![\tilde{e}_\alpha^{\tilde{X}}, \tilde{e}_\beta^{\tilde{X}}]\!]^S = C_{\alpha\beta}^\gamma\tilde{e}_\gamma^{\tilde{X}}, \qquad [\![\tilde{e}_\alpha^{\tilde{X}}, \bar{e}_\beta^{\tilde{X}}]\!]^S = [\![\bar{e}_\alpha^{\tilde{X}}, \bar{e}_\beta^{\tilde{X}}]\!]^S = 0,$$
$$\rho^S\left(\tilde{e}_\alpha^{\tilde{X}}\right) = (T\tilde{X})\left(\rho_\alpha^i\frac{\partial}{\partial x^i}\right), \qquad \rho^S\left(\bar{e}_\alpha^{\tilde{X}}\right) = (T\tilde{X})\left(\frac{\partial}{\partial y^\alpha}\right), \tag{7.6}$$

for all α and β. Now, we consider the map $\Theta^{\tilde{X}} : \mathcal{L}^\tau E \to \mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E)$ defined by

$$\Theta^{\tilde{X}}(a, v) = ((a, (T_{\tilde{X}(b)}(\tau^{\tau^*})^S)((T_b\tilde{X})(v))), (T_b\tilde{X})(v)),$$

for $(a, v) \in (\mathcal{L}^\tau E)_b \subseteq E_{\tau(b)} \times T_bE$, with $b \in E$. Note that $\tau^* \circ (\tau^{\tau^*})^S \circ \tilde{X} = \tau$ and, thus, $(a, (T_{\tilde{X}(b)}(\tau^{\tau^*})^S)((T_b\tilde{X})(v))) \in (\mathcal{L}^{\tau^*}E)_{(\tau^{\tau^*})^S(\tilde{X}(b))} \subseteq E_{\tau(b)} \times T_{\tau^{\tau^*}(\tilde{X}(b))}E^*$ which implies that

$$((a, (T_{\tilde{X}(b)}(\tau^{\tau^*})^S)((T_b\tilde{X})(v))), (T_b\tilde{X})(v)) \in \mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E)_{\tilde{X}(b)} \subseteq (\mathcal{L}^{\tau^*}E)_{(\tau^{\tau^*})^S(\tilde{X}(b))} \times T_{\tilde{X}(b)}S.$$

Furthermore, if $\{\tilde{T}_\alpha, V_\alpha\}$ is the local basis of $\Gamma(\mathcal{L}^\tau E)$ considered in remark 2.7 then a direct computation, using (7.5), proves that

$$\Theta^{\tilde{X}}(\tilde{T}_\alpha(a)) = \tilde{e}_\alpha^{\tilde{X}}(\tilde{X}(a)), \qquad \Theta^{\tilde{X}}(V_\alpha(a)) = \bar{e}_\alpha^{\tilde{X}}(\tilde{X}(a)), \tag{7.7}$$

for all $a \in \tau^{-1}(U)$. Therefore, from (2.33), (7.6) and (7.7), we conclude that the pair $(\Theta^{\tilde{X}}, \tilde{X})$ is an isomorphism between the Lie algebroids $\mathcal{L}^\tau E \to E$ and $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E) \to S$. Note that the local expression of $\Theta^{\tilde{X}}$ in the local coordinates $(x^i, y^\alpha; z^\alpha, v^\alpha)$ and $(x^i, y^\alpha; z_{\tilde{X}}^\alpha, v_{\tilde{X}}^\alpha)$ on $\mathcal{L}^\tau E$ and $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E)$ is just the identity, that is,

$$\Theta^{\tilde{X}}(x^i, y^\alpha; z^\alpha, v^\alpha) = (x^i, y^\alpha; z^\alpha, v^\alpha).$$

We recall that if E is a symplectic Lie algebroid with symplectic section Ω then the prolongation $\mathcal{L}^\tau E$ of E over the vector bundle projection $\tau : E \to M$ is a symplectic Lie algebroid with symplectic section $\Omega^{\mathbf{c}}$, the complete lift of Ω (see section 6).

Proposition 7.6. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a symplectic Lie algebroid with symplectic section Ω and X be a section of E. Denote by S the submanifold of E defined by $S = X(E)$, by*

$i : S \to E$ the canonical inclusion, by α_X the section of E^* given by $\alpha_X = i_X\Omega$ and by $\tau^S : S \to M$ the map defined by $\tau^S = \tau \circ i$, $\tau : E \to M$ being the vector bundle projection. Then, the Lie subalgebroid $(Id, Ti) : \mathcal{L}^{\tau^S}E \to \mathcal{L}^{\tau}E, i : S \to E$ of the symplectic Lie algebroid $((\mathcal{L}^{\tau}E, [\![\cdot,\cdot]\!]^{\tau}, \rho^{\tau}), \Omega^{\mathbf{c}})$ is Lagrangian if and only if α_X is a 1-cocycle.

Proof. Suppose that Y and Z are sections of E. Since Ω is a 2-cocycle, it follows that (see (2.1))

$$d\alpha_X(Y, Z) = \rho(X)(\Omega(Y, Z)) - \Omega(Y, [\![X, Z]\!]) - \Omega([\![X, Y]\!], Z). \tag{7.8}$$

On the other hand, using (6.3), we obtain that

$$\Omega^{\mathbf{c}}(Y^{\mathbf{c}} - [\![X, Y]\!]^{\mathbf{v}}, Z^{\mathbf{c}} - [\![X, Z]\!]^{\mathbf{v}}) = \Omega(Y, Z)^c - \Omega(Y, [\![X, Z]\!])^v - \Omega([\![X, Y]\!], Z)^v$$

which implies that (see (2.21))

$$\Omega^{\mathbf{c}}(Y^{\mathbf{c}} - [\![X, Y]\!]^{\mathbf{v}}, Z^{\mathbf{c}} - [\![X, Z]\!]^{\mathbf{v}}) \circ X = \rho(X)(\Omega(Y, Z)) - \Omega(Y, [\![X, Z]\!]) - \Omega([\![X, Y]\!], Z). \tag{7.9}$$

Therefore, from (7.8), (7.9) and taking into account that the rank of $\mathcal{L}^{\tau^S}E$ is $n = \frac{1}{2}\,\mathrm{rank}\,(\mathcal{L}^{\tau}E)$, we conclude that the Lie subalgebroid $(Id, Ti) : \mathcal{L}^{\tau^S}E \to \mathcal{L}^{\tau}E, i : S \to E$ is Lagrangian if and only if α_X is a cocycle (see example 7.5, (i)). $\square$

Remark 7.7. Let (M, Ω) be a symplectic manifold, $\tau_M : TM \to M$ be the standard symplectic Lie algebroid and X be a vector field on M. Then, the tangent bundle TM of M is a symplectic manifold with symplectic form the complete lift $\Omega^{\mathbf{c}}$ of Ω to TM (see remark 6.6). Moreover, using remark 7.2 and proposition 7.6, we deduce a well-known result: the submanifold $X(M)$ of TM is Lagrangian if and only if X is a locally Hamiltonian vector field of M.

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and denote by Ω_E the canonical symplectic section of the Lie algebroid $(\mathcal{L}^{\tau^*}E, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ (see sections 3.1 and 3.2) and by $\rho^*(TE^*) \to E$ the pull-back of the vector bundle $T\tau^* : TE^* \to TM$ over the anchor map $\rho : E \to TM$. As we know, $\rho^*(TE^*) = \mathcal{L}^{\tau^*}E$. Now, suppose that $\tilde{X} : E \to \rho^*(TE^*) = \mathcal{L}^{\tau^*}E$ is a section of $\rho^*(TE^*) \to E$. Then, $S = \tilde{X}(E)$ is a submanifold of $\mathcal{L}^{\tau^*}E$ and one may consider the Lie subalgebroid $(Id, Ti) : \mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E) \to \mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*}E), i : S \to \mathcal{L}^{\tau^*}E$ of the symplectic Lie algebroid $\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*}E) \to \mathcal{L}^{\tau^*}E$ (see example 7.5, (ii)). We remark that the symplectic section of $\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*}E) \to \mathcal{L}^{\tau^*}E$ is the complete lift $\Omega_E^{\mathbf{c}}$ of Ω_E to the prolongation of $\mathcal{L}^{\tau^*}E$ over the bundle projection $\tau^{\tau^*} : \mathcal{L}^{\tau^*}E \to E^*$.

On the other hand, let $A_E : \rho^*(TE^*) \to (\mathcal{L}^{\tau}E)^*$ be the canonical isomorphism between the vector bundles $\rho^*(TE^*) \to E$ and $(\mathcal{L}^{\tau}E)^* \to E$ considered in section 5 (see (5.5)) and $\alpha_{\tilde{X}}$ be the section of $(\mathcal{L}^{\tau}E)^* \to E$ given by

$$\alpha_{\tilde{X}} = A_E \circ \tilde{X}. \tag{7.10}$$

Then, we have the following result.

Proposition 7.8. *The Lie subalgebroid $(Id, Ti) : \mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*}E) \to \mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*}E), i : S \to \mathcal{L}^{\tau^*}E$ is Lagrangian if and only if the section $\alpha_{\tilde{X}}$ is 1-cocycle of the Lie algebroid $(\mathcal{L}^{\tau}E, [\![\cdot,\cdot]\!]^{\tau}, \rho^{\tau})$.*

Proof. Suppose that (x^i) are local coordinates on an open subset U of M and that $\{e_\alpha\}$ is the basis of the vector bundle $\tau^{-1}(U) \to U$. Then, we will use the following notation:

- $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ (respectively, $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$) is the local basis of $\Gamma(\mathcal{L}^\tau E)$ (respectively, $\Gamma(\mathcal{L}^{\tau^*} E)$) considered in remark 2.7 (respectively, section 3.1). $\{\tilde{T}^\alpha, \tilde{V}^\alpha\}$ (respectively, $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$) is the dual basis of $\{\tilde{T}_\alpha, \tilde{V}_\alpha\}$ (respectively, $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$).
- (x^i, y^α) (respectively, $(x^i, y_\alpha; z^\alpha, v_\alpha)$) are the corresponding local coordinates on E (respectively, $\mathcal{L}^{\tau^*} E$).
- ρ^i_α and $C^\gamma_{\alpha\beta}$ are the structure functions of E with respect to (x^i) and $\{e_\alpha\}$.
- $\Theta^{\tilde{X}} : \mathcal{L}^\tau E \to \mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*} E)$ is the isomorphism between the Lie algebroids $\mathcal{L}^\tau E \to E$ and $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*} E) \to S$ (over the map $\tilde{X} : E \to S = \tilde{X}(E)$) considered in example 7.5, (ii).

Then,

$$\left\{\tilde{e}^{\mathbf{c}}_\alpha, \bar{e}^{\mathbf{c}}_\alpha, \tilde{e}^{\mathbf{v}}_\alpha, \bar{e}^{\mathbf{v}}_\alpha\right\}$$

is a local basis of $\Gamma(\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*} E))$, where $\tilde{e}^{\mathbf{c}}_\alpha$ and $\bar{e}^{\mathbf{c}}_\alpha$ (respectively, $\tilde{e}^{\mathbf{v}}_\alpha$ and $\bar{e}^{\mathbf{v}}_\alpha$) are the complete (respectively, vertical) lifts of $\tilde{e}_\alpha$ and $\bar{e}_\alpha$ to $\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*} E)$. Moreover,

$$\{(\tilde{e}^\alpha)^{\mathbf{v}}, (\bar{e}^\alpha)^{\mathbf{v}}, (\tilde{e}^\alpha)^{\mathbf{c}}, (\bar{e}^\alpha)^{\mathbf{c}}\}$$

is the dual basis of $\left\{\tilde{e}^{\mathbf{c}}_\alpha, \bar{e}^{\mathbf{c}}_\alpha, \tilde{e}^{\mathbf{v}}_\alpha, \bar{e}^{\mathbf{v}}_\alpha\right\}$.

Now, assume that the local expression of the section $\tilde{X}$ is

$$\tilde{X}(x^i, y^\alpha) = (x^i, \tilde{X}^\alpha; y^\alpha, \tilde{X}'^\alpha). \tag{7.11}$$

Next, we consider the local basis $\left\{\tilde{e}^{\tilde{X}}_\alpha, \bar{e}^{\tilde{X}}_\alpha\right\}$ of $\Gamma\left(\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*} E)\right)$ given by (7.5). A direct computation, using (2.23), (2.25), (3.2) and (7.5), proves that

$$\begin{aligned}
\tilde{e}^{\tilde{X}}_\alpha(\tilde{X}(x^i, y^\gamma)) &= \tilde{e}^{\mathbf{c}}_\alpha(\tilde{X}(x^i, y^\gamma)) + \rho^i_\alpha(x^i)\frac{\partial \tilde{X}^\beta}{\partial x^i}_{|(x^i,y^\gamma)}\bar{e}^{\mathbf{c}}_\beta(\tilde{X}(x^i, y^\gamma)) \\
&\quad + C^\gamma_{\alpha\beta}(x^i)y^\beta\tilde{e}^{\mathbf{v}}_\gamma(\tilde{X}(x^i, y^\gamma)) + \rho^i_\alpha(x^i)\frac{\partial \tilde{X}'^\beta}{\partial x^i}_{|(x^i,y^\gamma)}\bar{e}^{\mathbf{v}}_\beta(\tilde{X}(x^i, y^\gamma)), \\
\bar{e}^{\tilde{X}}_\alpha(\tilde{X}(x^i, y^\gamma)) &= \frac{\partial \tilde{X}^\beta}{\partial y^\alpha}_{|(x^i,y^\gamma)}\bar{e}^{\mathbf{c}}_\beta(\tilde{X}(x^i, y^\gamma)) + \tilde{e}^{\mathbf{v}}_\alpha(\tilde{X}(x^i, y^\gamma)) \\
&\quad + \frac{\partial \tilde{X}'^\beta}{\partial y^\alpha}_{|(x^i,y^\gamma)}\bar{e}^{\mathbf{v}}_\beta(\tilde{X}(x^i, y^\gamma)).
\end{aligned} \tag{7.12}$$

Thus, if λ_E is the Liouville section of $\mathcal{L}^{\tau^*} E \to E$ then, from (6.8), (7.7) and (7.12), we obtain that

$$((Id, Ti) \circ \Theta^{\tilde{X}}, i \circ \tilde{X})^*\left(\lambda^{\mathbf{c}}_E\right) = \left(\tilde{X}'^\alpha + \tilde{X}^\beta C^\beta_{\alpha\gamma} y^\gamma\right)\tilde{T}^\alpha + \tilde{X}^\alpha \tilde{V}^\alpha.$$

Therefore, using (5.5), (7.10) and (7.11), it follows that

$$((Id, Ti) \circ \Theta^{\tilde{X}}, i \circ \tilde{X})^*\left(\lambda^{\mathbf{c}}_E\right) = \alpha_{\tilde{X}}.$$

Now, since $\Omega^{\mathbf{c}}_E = (-d^{\mathcal{L}^{\tau^*} E}\lambda_E)^{\mathbf{c}} = -d^{\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*} E)}\lambda^{\mathbf{c}}_E$ (see proposition 6.3), we have that

$$((Id, Ti) \circ \Theta^{\tilde{X}}, i \circ \tilde{X})^*\left(\Omega^{\mathbf{c}}_E\right) = -d^{\mathcal{L}^\tau E}\alpha_{\tilde{X}}. \tag{7.13}$$

Note that the pair $((Id, Ti) \circ \Theta^{\tilde{X}}, i \circ \tilde{X})$ is a morphism between the Lie algebroids $\mathcal{L}^\tau E \to E$ and $\mathcal{L}^{\tau^{\tau^*}}(\mathcal{L}^{\tau^*} E) \to \mathcal{L}^{\tau^*} E$.

Consequently, using (7.13) and since the rank of the vector bundle $\mathcal{L}^{(\tau^{\tau^*})^S}(\mathcal{L}^{\tau^*} E) \to S$ is $2n$, we deduce the result. □

8. Lagrangian submanifolds, Tulczyjew's triple and Euler–Lagrange (Hamilton) equations

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a symplectic Lie algebroid over a manifold M with symplectic section Ω. Then, the prolongation $\mathcal{L}^\tau E$ of E over the vector bundle projection $\tau : E \to M$ is a symplectic Lie algebroid with symplectic section $\Omega^{\mathbf{c}}$, the complete lift of Ω to $\mathcal{L}^\tau E$ (see theorem 6.5).

Definition 8.1. *Let S be a submanifold of the symplectic Lie algebroid E and $i : S \to E$ be the canonical inclusion. S is said to be Lagrangian if condition (7.3) holds and the corresponding Lie subalgebroid $(Id, Ti) : \mathcal{L}^{\tau^S} E \to \mathcal{L}^\tau E, i : S \to E$ of the symplectic Lie algebroid $(\mathcal{L}^\tau E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ is Lagrangian.*

Remark 8.2. Let M be a symplectic manifold with symplectic 2-form Ω and S be a submanifold of M. Denote by $i : S \to M$ the canonical inclusion and by $Ti : TS \to TM$ the tangent map to i. Then, Ti is an injective inmersion and TS is a submanifold of TM, the standard Lie algebroid $\tau_M : TM \to M$ is symplectic, the prolongation $\mathcal{L}^{\tau_M}(TM)$ of $\tau_M : TM \to M$ over τ_M is the standard Lie algebroid $\tau_{TM} : T(T(M)) \to TM$ and $\Omega^{\mathbf{c}}$ is the usual complete lift of Ω to TM (see remark 6.4). Moreover, the Lie subalgebroid $(Id, T(Ti)) : \mathcal{L}^{\tau_M^{TS}}(TM) \to \mathcal{L}^{\tau_M}(TM) = T(TM), Ti : TS \to TM$ is just the standard Lie algebroid $\tau_{TS} : T(TS) \to TS$. Thus, TS is a Lagrangian submanifold of TM in the sense of definition 8.1 if and only if TS is a Lagrangian submanifold of the symplectic manifold $(TM, \Omega^{\mathbf{c}})$ in the usual sense (see remark 7.2). On the other hand, we have that

$$(Ti)^*(\Omega^{\mathbf{c}}) = (i^*(\Omega))^{\mathbf{c}}, \tag{8.1}$$

where $(i^*(\Omega))^{\mathbf{c}}$ is the usual complete lift of the 2-form $i^*\Omega$ to TS. From (8.1), it follows that TS is a Lagrangian submanifold of the symplectic manifold $(TM, \Omega^{\mathbf{c}})$ in the usual sense if and only if S is a Lagrangian submanifold of the symplectic manifold (M, Ω) in the usual sense. Therefore, we conclude that the following conditions are equivalent:

(i) S is a Lagrangian submanifold of the symplectic manifold (M, Ω) in the usual sense.
(ii) TS is a Lagrangian submanifold of the symplectic manifold $(TM, \Omega^{\mathbf{c}})$ in the usual sense.
(iii) TS is a Lagrangian submanifold of TM in the sense of definition 8.1.

From proposition 7.6, we deduce

Corollary 8.3. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a symplectic Lie algebroid over a manifold M with symplectic section Ω and X be a section of E. If α_X is the section of E^* given by*

$$\alpha_X = i_X\Omega$$

and α_X is a 1-cocycle of E then $S = X(M)$ is a Lagrangian submanifold of E.

If $(E, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid over a manifold M, we will denote by $\rho^*(TE^*) \to E$ the pull-back of the vector bundle $T\tau^* : TE^* \to TM$ over the anchor map $\rho : E \to TM$, by $\mathcal{L}^{\tau^*}E$ the prolongation of E over the vector bundle projection $\tau^* : E^* \to M$ and by $A_E : \rho^*(TE^*) \to (\mathcal{L}^\tau E)^*$ the isomorphism of vector bundles considered in section 5.

Using proposition 7.8, we have the following result

Corollary 8.4. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $\tilde{X}$ be a section of the vector bundle $\rho^*(TE^*) \to E$. If $\alpha_{\tilde{X}}$ is the section of $(\mathcal{L}^\tau E)^* \to E$ given by*

$$\alpha_{\tilde{X}} = A_E \circ \tilde{X}$$

and $\alpha_{\tilde{X}}$ is a 1-cocycle of Lie algebroid $(\mathcal{L}^\tau E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ then $S = \tilde{X}(E)$ is a Lagrangian submanifold of the symplectic Lie algebroid $\mathcal{L}^{\tau^*}E$.

Now, let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold M and $H : E^* \to \mathbb{R}$ be a Hamiltonian function. If Ω_E is the canonical symplectic section of $\mathcal{L}^{\tau^*}E$, then there exists a unique section ξ_H of $\mathcal{L}^{\tau^*}E \to E^*$ such that

$$i_{\xi_H}\Omega_E = d^{\mathcal{L}^{\tau^*}E}H.$$

Moreover, from corollary 8.3, we deduce that $S_H = \xi_H(E^*)$ is a Lagrangian submanifold of $\mathcal{L}^{\tau^*}E$.

On the other hand, it is clear that there exists a bijective correspondence Ψ_H between the set of curves in S_H and the set of curves in E^*. In fact, if $c : I \to E^*$ is a curve in E^* then the corresponding curve in S_H is $\xi_H \circ c : I \to S_H$.

A curve γ in S_H,

$$\gamma : I \to S_H \subseteq \mathcal{L}^{\tau^*}E \subseteq E \times TE^*, \qquad t \to (\gamma_1(t), \gamma_2(t))$$

is said to be *admissible* if the curve $\gamma_2 : I \to TE^*, t \to \gamma_2(t)$, is a tangent lift, that is,

$$\gamma_2(t) = \dot{c}(t),$$

where $c : I \to E^*$ is the curve in E^* given by $\tau_{E^*} \circ \gamma_2$, $\tau_{E^*} : TE^* \to E^*$ being the canonical projection.

Theorem 8.5. *Under the bijection Ψ_H, the admissible curves in the Lagrangian submanifold S_H correspond with the solutions of the Hamilton equations for H.*

Proof. Let $\gamma : I \to S_H \subseteq \mathcal{L}^{\tau^*}E \subseteq E \times TE^*$ be an admissible curve in S_H,

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)),$$

for all t. Then, $\gamma_2(t) = \dot{c}(t)$, for all t, where $c : I \to E^*$ is the curve in E^* given by $c = \tau_{E^*} \circ \gamma_2$.

Now, since ξ_H is a section of the vector bundle $\tau^{\tau^*} : \mathcal{L}^{\tau^*}E \to E^*$ and $\gamma(I) \subseteq S_H = \xi_H(E^*)$, it follows that

$$\xi_H(c(t)) = \gamma(t), \qquad \text{for all} \quad t \tag{8.2}$$

that is, $c = \Psi_H(\gamma)$. Thus, from (8.2), we obtain that

$$\rho^{\tau^*}(\xi_H) \circ c = \gamma_2 = \dot{c},$$

that is, c is an integral curve of the vector field $\rho^{\tau^*}(\xi_H)$ and, therefore, c is a solution of the Hamilton equations associated with H (see section 3.3).

Conversely, assume that $c : I \to E^*$ is a solution of the Hamilton equations associated with H, that is, c is an integral curve of the vector field $\rho^{\tau^*}(\xi_H)$ or, equivalently,

$$\rho^{\tau^*}(\xi_H) \circ c = \dot{c}. \tag{8.3}$$

Then, $\gamma = \xi_H \circ c$ is a curve in S_H and, from (8.3), we deduce that γ is admissible. □

Next, suppose that $L : E \to \mathbb{R}$ is a Lagrangian function. Then, from corollary 8.4, we obtain that $S_L = \left(A_E^{-1} \circ d^{\mathcal{L}^{\tau}E}L\right)(E)$ is a Lagrangian submanifold of the symplectic Lie algebroid $\mathcal{L}^{\tau^*}E$.

On the other hand, we have a bijective correspondence Ψ_L between the set of curves in S_L and the set of curves in E. In fact, if $\gamma : I \to S_L$ is a curve in S_L then there exists a unique curve $c : I \to E$ in E such that

$$A_E(\gamma(t)) = (d^{\mathcal{L}^{\tau}E}L)(c(t)), \qquad \text{for all} \quad t.$$

Note that

$$pr_1(\gamma(t)) = (\tau^{\tau})^*(A_E(\gamma(t))) = (\tau^{\tau})^*((d^{\mathcal{L}^{\tau}E}L)(c(t))) = c(t), \qquad \text{for all} \quad t,$$

where $pr_1 : \mathcal{L}^{\tau^*}E \subseteq E \times TE^* \to E$ is the canonical projection on the first factor and $(\tau^\tau)^* : (\mathcal{L}^\tau E)^* \to E$ is the vector bundle projection. Thus,

$$\gamma(t) = (c(t), \gamma_2(t)) \in S_L \subseteq \mathcal{L}^{\tau^*}E \subseteq E \times TE^*, \qquad \text{for all} \quad t.$$

A curve γ in S_L

$$\gamma : I \to S_L \subseteq \mathcal{L}^{\tau^*}E \subseteq E \times TE^*, \qquad t \to (c(t), \gamma_2(t))$$

is said to be *admissible* if the curve

$$\gamma_2 : I \to TE^*, \qquad t \to \gamma_2(t)$$

is a tangent lift, that is, $\gamma_2(t) = \dot{c}^*(t)$, where $c^* : I \to E^*$ is the curve in E^* given by $c^* = \tau_{E^*} \circ \gamma_2$.

Theorem 8.6. *Under the bijection Ψ_L, the admissible curves in the Lagrangian submanifold S_L correspond with the solutions of the Euler–Lagrange equations for L.*

Proof. Suppose that (x^i) are local coordinates on M and that $\{e_\alpha\}$ is a local basis of $\Gamma(E)$. Denote by (x^i, y^α) (respectively, (x^i, y_α) and $(x^i, y_\alpha; z^\alpha, v_\alpha)$) the corresponding coordinates on E (respectively, E^* and $\mathcal{L}^{\tau^*}E$). Then, using (2.30), (2.32) and (5.5), it follows that the submanifold S_L is characterized by the following equations,

$$y_\alpha = \frac{\partial L}{\partial y^\alpha}, \qquad z^\alpha = y^\alpha, \qquad v_\alpha = \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} \frac{\partial L}{\partial y^\gamma} y^\beta, \tag{8.4}$$

for all $\alpha \in \{1, \ldots, n\}$.

Now, let $\gamma : I \to S_L$ be an admissible curve in S_L

$$\gamma(t) = (c(t), \gamma_2(t)) \in S_L \subseteq \mathcal{L}^{\tau^*}E \subseteq E \times TE^*, \qquad \text{for all} \quad t$$

and denote by $c^* : I \to E^*$ the curve in E^* satisfying

$$\gamma_2(t) = \dot{c}^*(t), \qquad \text{for all} \quad t, \tag{8.5}$$

i.e.,

$$c^*(t) = \tau_{E^*}(\gamma_2(t)), \qquad \text{for all} \quad t. \tag{8.6}$$

If the local expressions of γ and c are

$$\gamma(t) = (x^i(t), y_\alpha(t); z^\alpha(t), v_\alpha(t)), \qquad c(t) = (x^i(t), y^\alpha(t)),$$

then we have that

$$y^\alpha(t) = z^\alpha(t), \qquad \text{for all} \quad \alpha. \tag{8.7}$$

Moreover, from (8.5) and (8.6), we deduce that

$$c^*(t) = (x^i(t), y_\alpha(t)), \qquad \gamma_2(t) = \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\partial}{\partial x^i}_{|c^*(t)} + \frac{\mathrm{d}y_\alpha}{\mathrm{d}t} \frac{\partial}{\partial y_\alpha}_{|c^*(t)}. \tag{8.8}$$

Thus,

$$v_\alpha(t) = \frac{\mathrm{d}y_\alpha}{\mathrm{d}t}, \qquad \text{for all} \quad \alpha. \tag{8.9}$$

Therefore, using (8.4), (8.7), (8.8), (8.9) and the fact that $\rho(c(t)) = (T\tau^*)(\gamma_2(t))$, it follows that

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \rho^i_\alpha y^\alpha, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma},$$

for all i and α, that is, c is a solution of the Euler–Lagrange equations for L.

Conversely, let $c : I \to E$ be a solution of the Euler–Lagrange equations for L and $\gamma : I \to S_L$ be the corresponding curve in S_L,

$$c = \Psi_L(\gamma).$$

Suppose that

$$\gamma(t) = (c(t), \gamma_2(t)) \in \mathcal{L}^{\tau^*}E \subseteq E \times TE^*, \qquad \text{for all} \quad t$$

and denote by $c^* : I \to E^*$ the curve in E^* given by

$$c^* = \tau_{E^*} \circ \gamma_2.$$

If the local expressions of γ and c are

$$\gamma(t) = (x^i(t), y_\alpha(t); z^\alpha(t), v_\alpha(t)), \qquad c(t) = (x^i(t), y^\alpha(t)),$$

then

$$y^\alpha(t) = z^\alpha(t), \qquad \text{for all} \quad \alpha \tag{8.10}$$

and the local expressions of c^* and γ_2 are

$$c^*(t) = (x^i(t), y_\alpha(t)), \qquad \gamma_2(t) = z^\alpha(t)\rho^i_\alpha(x^j(t))\frac{\partial}{\partial x^i}_{|c^*(t)} + v_\alpha(t)\frac{\partial}{\partial y_\alpha}_{|c^*(t)}.$$

Thus, using (8.4) and the fact that c is a solution of the Euler–Lagrange equations for L, we deduce that

$$\gamma_2(t) = \dot{c}^*(t), \qquad \text{for all} \quad t,$$

which implies that γ is admissible. □

Now, assume that the Lagrangian function $L : E \to \mathbb{R}$ is hyperregular and denote by ω_L, E_L and ξ_L the Poincaré–Cartan 2-section, the energy function and the Euler–Lagrange section associated with L, respectively. Then, ω_L is a symplectic section of the Lie algebroid $(\mathcal{L}^\tau E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ and

$$i_{\xi_L}\omega_L = d^{\mathcal{L}^\tau E}E_L,$$

(see section 2.2.2). Moreover, from corollary 8.3, we deduce that

$$S_{\xi_L} = \xi_L(E)$$

is a Lagrangian submanifold of the symplectic Lie algebroid $\mathcal{L}^\tau E$.

On the other hand, it is clear that there exists a bijective correspondence $\Psi_{S_{\xi_L}}$ between the set of curves in S_{ξ_L} and the set of curves in E.

A curve γ in S_{ξ_L}

$$\gamma : I \to S_{\xi_L} \subseteq \mathcal{L}^\tau E \subseteq E \times TE, \qquad t \to (\gamma_1(t), \gamma_2(t)),$$

is said to be *admissible* if the curve

$$\gamma_2 : I \to TE, \qquad t \to \gamma_2(t)$$

is a tangent lift, that is,

$$\gamma_2(t) = \dot{c}(t), \qquad \text{for all} \quad t,$$

where $c : I \to E$ is the curve in E defined by $c = \tau_E \circ \gamma_2$, $\tau_E : TE \to E$ being the canonical projection.

Theorem 8.7. *If the Lagrangian L is hyperregular then under the bijection $\psi_{S_{\xi_L}}$ the admissible curves in the Lagrangian submanifold S_{ξ_L} correspond with the solutions of the Euler–Lagrange equations for L.*

Proof. Let $\gamma : I \to S_{\xi_L} \subseteq \mathcal{L}^\tau E \subseteq E \times TE$ be an admissible curve in S_{ξ_L}

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)), \qquad \text{for all} \quad t.$$

Then,

$$\gamma_2(t) = \dot{c}(t), \qquad \text{for all} \quad t,$$

where $c : I \to E$ is the curve in E given by $c = \tau_E \circ \gamma_2$.

Now, since ξ_L is a section of the vector bundle $\tau^\tau : \mathcal{L}^\tau E \to E$ and $\gamma(I) \subseteq S_{\xi_L} = \xi_L(E)$, it follows that

$$\xi_L(c(t)) = \gamma(t), \qquad \text{for all} \quad t \tag{8.11}$$

that is, $c = \Psi_{S_{\xi_L}}(\gamma)$.

Thus, from (8.11), we obtain that

$$\rho^\tau(\xi_L) \circ c = \gamma_2 = \dot{c},$$

that is, c is an integral curve of the vector field $\rho^\tau(\xi_L)$ and, therefore, c is a solution of the Euler–Lagrange equations associated with L (see section 2.2.2).

Conversely, assume that $c : I \to E$ is a solution of the Euler–Lagrange equations associated with L, that is, $c : I \to E$ is an integral curve of the vector field $\rho^\tau(\xi_L)$ or, equivalently,

$$\rho^\tau(\xi_L) \circ c = \dot{c}. \tag{8.12}$$

Then, $\gamma = \xi_L \circ c$ is a curve in S_{ξ_L} and, from (8.12), we deduce that γ is admissible. □

If $L : E \to \mathbb{R}$ is hyperregular then the Legendre transformation $Leg_L : E \to E^*$ associated with L is a global diffeomorphism and we may consider the Lie algebroid isomorphism $\mathcal{L}Leg_L : \mathcal{L}^\tau E \to \mathcal{L}^{\tau^*} E$ given by (3.25) and the Hamiltonian function $H : E^* \to \mathbb{R}$ defined by $H = E_L \circ Leg_L^{-1}$ (see section 3.6).

Thus, we have:

- The Lagrangian submanifolds S_L and S_H of the symplectic Lie algebroid $\mathcal{L}^{\tau^*} E$.
- The Lagrangian submanifold S_{ξ_L} of the symplectic Lie algebroid $\mathcal{L}^\tau E$.

Theorem 8.8. *If the Lagrangian function $L : E \to \mathbb{R}$ is hyperregular and $H : E^* \to \mathbb{R}$ is the corresponding Hamiltonian function then the Lagrangian submanifolds S_L and S_H are equal and*

$$\mathcal{L}Leg_L\left(S_{\xi_L}\right) = S_L = S_H. \tag{8.13}$$

Proof. Using (3.32), we obtain that

$$A_E \circ \xi_H \circ Leg_L = A_E \circ \mathcal{L}Leg_L \circ \xi_L. \tag{8.14}$$

Now, suppose that (x^i) are local coordinates in M and that $\{e_\alpha\}$ is a local basis of $\Gamma(E)$. Denote by (x^i, y^α) the corresponding coordinates on E and by $(x^i, y^\alpha; z^\alpha, v^\alpha)$ (respectively, $(x^i, y_\alpha; z^\alpha, v_\alpha)$) the corresponding ones on $\mathcal{L}^\tau E$ (respectively, $\mathcal{L}^{\tau^*} E$). Then, from (2.42), (3.26) and (5.5), we deduce that

$$(A_E \circ \mathcal{L}Leg_L \circ \xi_L)(x^i, y^\alpha) = \left(x^i, y^\alpha; \rho^i_\alpha \frac{\partial L}{\partial x^i}, \frac{\partial L}{\partial y^\alpha}\right).$$

Thus, from (2.36), it follows that

$$(A_E \circ \mathcal{L}Leg_L \circ \xi_L)(x^i, y^\alpha) = d^{\mathcal{L}^\tau E} L(x^i, y^\alpha),$$

that is (see (8.14)),

$$A_E \circ \xi_H \circ Leg_L = d^{\mathcal{L}^\tau E} L.$$

Therefore, $S_L = S_H$. On the other hand, using (3.32), we obtain that (8.13) holds. □

9. An application: Lagrangian submanifolds in prolongations of Atiyah algebroids and Lagrange (Hamilton)–Poincaré equations

9.1. Prolongations of the Atiyah algebroid associated with a principal bundle

Let $\pi : Q \to M$ be a principal bundle with structural group G, $\phi : G \times Q \to Q$ be the free action of G on Q and $\tau_Q|G : TQ/G \to M$ be the Atiyah algebroid associated with $\pi : Q \to M$ (see section 2.1.3).

The tangent action ϕ^T of G on TQ is free and thus, TQ is the total space of a principal bundle over TQ/G with structural group G. The canonical projection $\pi_T : TQ \to TQ/G$ is just the bundle projection.

Now, let $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ be the prolongation of the Atiyah algebroid $\tau_Q|G : TQ/G \to M$ by the vector bundle projection $\tau_Q|G : TQ/G \to M$, and denote by $(\phi^T)^{T^*} : G \times T^*(TQ) \to T^*(TQ)$ the cotangent lift of the tangent action $\phi^T : G \times TQ \to TQ$.

Theorem 9.1. *Let $\pi : Q \to M$ be a principal bundle with structural group G and $\tau_Q|G : TQ/G \to M$ be the Atiyah algebroid associated with the principal bundle. Then:*

(i) The Lie algebroid $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ and the Atiyah algebroid associated with the principal bundle $\pi_T : TQ \to TQ/G$ are isomorphic.
(ii) The dual vector bundle to $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ is isomorphic to the quotient vector bundle of $\pi_{TQ} : T^*(TQ) \to TQ$ by the action $(\phi^T)^{T^*}$ of G on $T^*(TQ)$.

Proof. (i) The Atiyah algebroid associated with the principal bundle $\pi_T : TQ \to TQ/G$ is the quotient vector bundle $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ of $\tau_{TQ} : T(TQ) \to TQ$ by the action $(\phi^T)^T$ of G on $T(TQ)$.

On the other hand, we have that the fibre of $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ over $[u_q] \in TQ/G$ is the subspace of $(TQ/G)_{\pi(q)} \times T_{[u_q]}(TQ/G)$ defined by

$$\begin{aligned}\mathcal{L}^{(\tau_Q|G)}(TQ/G)_{[u_q]} = \big\{&\big([v_q], X_{[u_q]}\big) \in (TQ/G)_{\pi(q)} \times T_{[u_q]}(TQ/G)/\\ &(T_q\pi)(v_q) = \big(T_{[u_q]}(\tau_Q|G)\big)\big(X_{[u_q]}\big)\big\}.\end{aligned}$$

Now, we define the morphism $(\pi_T \circ T\tau_Q, T\pi_T)$ between the vector bundles $\tau_{TQ} : T(TQ) \to TQ$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$ over the map $\pi_T : TQ \to TQ/G$ as follows,

$$(\pi_T \circ T\tau_Q, T\pi_T)\big(X_{u_q}\big) = \big(\pi_T\big(\big(T_{u_q}\tau_Q\big)\big(X_{u_q}\big)\big), \big(T_{u_q}\pi_T\big)\big(X_{u_q}\big)\big) \qquad (9.1)$$

for $X_{u_q} \in T_{u_q}(TQ)$, with $u_q \in T_qQ$.

Since the following diagram

$$\begin{array}{ccc} TQ & \xrightarrow{\tau_Q} & Q \\ \pi_T \big\downarrow & & \big\downarrow \pi \\ TQ/G & \xrightarrow{\tau_Q|G} & M = Q/G \end{array}$$

is commutative, one deduces that

$$(\pi_T \circ T\tau_Q, T\pi_T)\big(X_{u_q}\big) \in \mathcal{L}^{(\tau_Q|G)}(TQ/G)_{[u_q]}$$

and, thus, the map $(\pi_T \circ T\tau_Q, T\pi_T)$ is well-defined.

Next, we will proceed in two steps.

First step. *We will prove that the map* $(\pi_T \circ T\tau_Q, T\pi_T)$ *induces an isomorphism* $(\widetilde{\pi_T \circ T\tau_Q, T\pi_T})$ *between the vector bundles* $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ *and* $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$.

It is clear that

$$(\pi_T \circ T\tau_Q, T\pi_T)_{|T_{u_q}(TQ)} : T_{u_q}(TQ) \to \mathcal{L}^{(\tau_Q|G)}(TQ/G)_{[u_q]}$$

is linear. In addition, this map is injective. In fact, if $(\pi_T \circ T\tau_Q, T\pi_T)\big(X_{u_q}\big) = 0$ then $\big(T_{u_q}\pi_T\big)\big(X_{u_q}\big) = 0$ and there exists $\xi \in \mathfrak{g} \cong T_eG$, e being the identity element of G, such that

$$X_{u_q} = \big(T_e(\phi^T)_{u_q}\big)(\xi), \tag{9.2}$$

where $(\phi^T)_{u_q} : G \to TQ$ is defined by

$$(\phi^T)_{u_q}(g) = (\phi^T)_g(u_q) = (T_q\phi_g)(u_q), \qquad \text{for} \quad g \in G.$$

Therefore, using that $\pi_T\big(\big(T_{u_q}\tau_Q\big)\big(X_{u_q}\big)\big) = 0$, and hence $\big(T_{u_q}\tau_Q\big)\big(X_{u_q}\big) = 0$, we have that

$$0 = T_e\big(\tau_Q \circ (\phi^T)_{u_q}\big)(\xi) = (T_e\phi_q)(\xi),$$

$\phi_q : G \to Q$ being the injective immersion given by

$$\phi_q(g) = \phi_g(q) = \phi(g, q), \qquad \text{for} \quad g \in G.$$

Consequently, $\xi = 0$ and $X_{u_q} = 0$ (see (9.2)).

We have proved that the linear map $(\pi_T \circ T\tau_Q, T\pi_T)_{|T_{u_q}(TQ)}$ is injective which implies that $(\pi_T \circ T\tau_Q, T\pi_T)_{|T_{u_q}(TQ)}$ is a linear isomorphism (note that the dimensions of the spaces $T_{u_q}(TQ)$ and $\mathcal{L}^{(\tau_Q|G)}(TQ/G)_{[u_q]}$) are equal).

Furthermore, since the following diagram

$$\begin{tikzcd}
 & TQ \arrow[dl, "\pi_T"'] \arrow[rr, "\tau_Q"] \arrow[dd, "\phi^T_g = T\phi_g"] & & Q \arrow[dd, "\phi_g"] \\
TQ/G & & & \\
 & TQ \arrow[ul, "\pi_T"] \arrow[rr, "\tau_Q"] & & Q
\end{tikzcd}$$

is commutative, we deduce that $(\pi_T \circ T\tau_Q, T\pi_T)$ induces a morphism $(\widetilde{\pi_T \circ T\tau_Q, T\pi_T})$ between the vector bundles $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$ in such a way that the following diagram

$$\begin{tikzcd}
T(TQ) \arrow[r, "(\pi_T \circ T\tau_Q, T\pi_T)"] \arrow[d, "(\pi_T)_T"'] & \mathcal{L}^{(\tau_Q|G)}(TQ/G) \\
T(TQ)/G \arrow[ur, "(\widetilde{\pi_T \circ T\tau_Q, T\pi_T})"'] &
\end{tikzcd}$$

is commutative, where $(\pi_T)_T : T(TQ) \to T(TQ)/G$ is the canonical projection. In addition, if $u_q \in T_qQ$ then, since the map

$$((\pi_T)_T)_{|T_{u_q}(TQ)} : T_{u_q}(TQ) \to (T(TQ)/G)_{[u_q]}$$

is a linear isomorphism, we conclude that $(\pi_T \circ \widetilde{T\tau_Q}, T\pi_T)$ is a isomorphism between the vector bundles $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{\tau_Q|G}(TQ/G) \to TQ/G$.

Second step. *We will prove that the map*

$$(\pi_T \circ \widetilde{T\tau_Q}, T\pi_T) : T(TQ)/G \to \mathcal{L}^{(\tau_Q|G)}(TQ/G)$$

is an isomorphism between the Atiyah algebroid associated with the principal bundle $\pi_T : TQ \to TQ/G$ *and* $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$.

Let $A : TQ \to \mathfrak{g}$ be a (principal) connection on the principal bundle $\pi : Q \to M = Q/G$. We choose a local trivialization of $\pi : Q \to M$ to be $U \times G$, where U is an open subset of M. Then, G acts on $U \times G$ as follows,

$$\phi(g, (x, g')) = (x, gg'), \qquad \text{for} \quad g \in G \quad \text{and} \quad (x, g') \in U \times G. \tag{9.3}$$

Assume that there are local coordinates (x^i) on U and that $\{\xi_a\}$ is a basis of $\mathfrak{g}$. Denote by $\left\{\xi_a^L\right\}$ the corresponding left-invariant vector fields on G and suppose that

$$A\left(\frac{\partial}{\partial x^i}_{|(x,e)}\right) = A_i^a(x)\xi_a,$$

for $i \in \{1, \ldots, m\}$ and $x \in U$. Then, as we know (see remark 2.4), the vector fields on $U \times G$

$$\left\{e_i = \frac{\partial}{\partial x^i} - A_i^a \xi_a^L, e_b = \xi_b^L\right\}$$

define a local basis $\{e_i', e_b'\}$ of $\Gamma(TQ/G)$, such that $\pi_T \circ e_i = e_i' \circ \pi$, and $\pi_T \circ e_b = e_b' \circ \pi$. Thus, one may consider the local coordinates $(x^i, \dot{x}^i, \bar{v}^b)$ on TQ/G induced by the local basis $\{e_i', e_b'\}$ and the corresponding local basis $\{\tilde{T}_i, \tilde{T}_b, \tilde{V}_i, \tilde{V}_b\}$ of $\Gamma(\mathcal{L}^{(\tau_Q|G)}(TQ/G))$. From (2.20) and (2.31), we have that

$$\begin{aligned} \tilde{T}_i[u_q] &= \left(e_i'(\pi(q)), \frac{\partial}{\partial x^i}_{|[u_q]}\right), \qquad & \tilde{T}_b[u_q] &= (e_b'(\pi(q)), 0), \\ \tilde{V}_i[u_q] &= \left(0, \frac{\partial}{\partial \dot{x}^i}_{|[u_q]}\right), & \tilde{V}_b[u_q] &= \left(0, \frac{\partial}{\partial \bar{v}^b}_{|[u_q]}\right), \end{aligned} \tag{9.4}$$

for $u_q \in T_q Q$, with $q \in Q$.

On the other hand, using the left translations by elements of G, one may identify the tangent bundle to G, TG, with the product manifold $G \times \mathfrak{g}$ and, under this identification, the tangent action of G on $T(U \times G) \cong TU \times TG \cong TU \times (G \times \mathfrak{g})$ is given by (see (9.3))

$$\phi^T(g, (v_x, (g', \xi))) = (v_x, (gg', \xi)),$$

for $g \in G$, $v_x \in T_x U$ and $(g', \xi) \in G \times \mathfrak{g}$. Therefore, $T(U \times G)/G \cong TU \times \mathfrak{g}$ and the vector fields on $T(U \times G) \cong TU \times (G \times \mathfrak{g})$ defined by

$$\begin{aligned} \tilde{X}_i &= \frac{\partial}{\partial x^i} - A_i^a \xi_a^L, \qquad & \tilde{X}_b &= \xi_b^L, \\ \bar{X}_i &= \frac{\partial}{\partial \dot{x}^i}, & \bar{X}_b &= \frac{\partial}{\partial \bar{v}^b}, \end{aligned} \tag{9.5}$$

are ϕ^T-invariant and they define a (local) basis of $\Gamma(T(TQ)/G)$. Moreover, it follows that

$$\begin{aligned} T\tau_Q \circ \tilde{X}_i &= e_i \circ \tau_Q, \qquad & T\tau_Q \circ \tilde{X}_b &= e_b \circ \tau_Q, \\ T\tau_Q \circ \bar{X}_i &= 0, & T\tau_Q \circ \bar{X}_b &= 0, \\ T\pi_T \circ \tilde{X}_i &= \frac{\partial}{\partial x^i} \circ \pi_T, & T\pi_T \circ \tilde{X}_b &= 0, \\ T\pi_T \circ \bar{X}_i &= \frac{\partial}{\partial \dot{x}^i} \circ \pi_T, & T\pi_T \circ \bar{X}_b &= \frac{\partial}{\partial \bar{v}^b} \circ \pi_T. \end{aligned}$$

This implies that

$$\begin{aligned} &(\pi_T \circ T\tau_Q, T\pi_T) \circ \tilde{X}_i = \tilde{T}_i \circ \pi_T, &&(\pi_T \circ T\tau_Q, T\pi_T) \circ \tilde{X}_b = \tilde{T}_b \circ \pi_T, \\ &(\pi_T \circ T\tau_Q, T\pi_T) \circ \bar{X}_i = \tilde{V}_i \circ \pi_T, &&(\pi_T \circ T\tau_Q, T\pi_T) \circ \bar{X}_b = \tilde{V}_b \circ \pi_T. \end{aligned} \tag{9.6}$$

Furthermore, if c_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_a\}$, $B : TQ \oplus TQ \to \mathfrak{g}$ is the curvature of A and

$$B\left(\frac{\partial}{\partial x^i}_{|(x,e)}, \frac{\partial}{\partial x^j}_{|(x,e)}\right) = B_{ij}^a(x)\xi_a, \qquad \text{for} \quad x \in U,$$

then, a direct computation proves that,

$$[\tilde{X}_i, \tilde{X}_j] = -B_{ij}^a \tilde{X}_a, \qquad [\tilde{X}_i, \tilde{X}_a] = c_{ab}^c A_i^b \tilde{X}_c, \qquad [\tilde{X}_a, \tilde{X}_b] = c_{ab}^c \tilde{X}_c,$$

and the rest of the Lie brackets of the vector fields $\{\tilde{X}_i, \tilde{X}_a, \bar{X}_i, \bar{X}_a\}$ are zero. Thus, from (2.20) and (2.33), we conclude that $(\pi_T \circ \widetilde{T\tau_Q}, T\pi_T)$ is a Lie algebroid isomorphism.

(ii) It follows using (i) and the results of section 2.1.3 (see example 2.3, (b)). □

Remark 9.2. As we know $\mathcal{L}^{\tau_Q}(TQ) \cong T(TQ)$ (see remark 2.6). In addition, if $\{X_i, X_b\}$ is the local basis of $\Gamma(TQ) = \mathfrak{X}(Q)$ given by

$$X_i = \frac{\partial}{\partial x^i} - A_i^a \xi_a^L, \qquad X_b = \xi_b^L$$

then the corresponding (local) basis of $\Gamma(\mathcal{L}^{\tau_Q}(TQ)) \cong \Gamma(T(TQ)) = \mathfrak{X}(TQ)$ is $\{\tilde{X}_i, \tilde{X}_b, \bar{X}_i, \bar{X}_b\}$, where $\tilde{X}_i, \tilde{X}_b, \bar{X}_i$ and $\bar{X}_b$ are the local vector fields on $T(U \times G) \cong TU \times (G \times \mathfrak{g})$ defined by (9.5). One may deduce this result using (2.31), the fact that the anchor map of $\tau_Q : TQ \to Q$ is the identity and the following equalities

$$X_i^v = \frac{\partial}{\partial \dot{x}_i}, \qquad X_b^v = \frac{\partial}{\partial \bar{v}^b},$$

where X_i^v (respectively X_b^v) is the vertical lift of X_i (respectively, X_b).

If $\pi : Q \to M$ is a principal bundle with structural group G and $\phi : G \times Q \to Q$ is the free action of G on Q then, as we know (see section 2.1.3), the dual vector bundle to the Atiyah algebroid $\tau_Q|G : TQ/G \to M$ is isomorphic to the quotient vector bundle of the cotangent bundle $\pi_Q : T^*Q \to Q$ by the cotangent action ϕ^{T^*} of G on T^*Q, that is, the vector bundles $(\tau_Q|G)^* : (TQ/G)^* \to M$ and $\pi_Q|G : T^*Q/G \to M$ are isomorphic. Since ϕ^{T^*} is a free action, T^*Q is the total space of a principal bundle over T^*Q/G with structural group G. The canonical projection $\pi_{T^*} : T^*Q \to T^*Q/G$ is just the bundle projection.

Now, denote by $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ the prolongation of the Atiyah algebroid $\tau_Q|G : TQ/G \to M$ by the vector bundle projection $\pi_Q|G : T^*Q/G \to M$ and by $(\phi^{T^*})^{T^*} : G \times T^*(T^*Q) \to T^*(T^*Q)$ the cotangent lift of the cotangent action $\phi^{T^*} : G \times T^*Q \to T^*Q$.

Theorem 9.3. *Let $\pi : Q \to M$ be a principal bundle with structural group G and $\tau_Q|G : TQ/G \to M$ be the Atiyah algebroid associated with the principal bundle. Then:*

(i) The Lie algebroid $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ and the Atiyah algebroid associated with the principal bundle $\pi_{T^*} : T^*Q \to T^*Q/G$ are isomorphic.

(ii) The dual vector bundle to $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ is isomorphic to the quotient vector bundle of $\pi_{T^*Q} : T^*(T^*Q) \to T^*Q$ by the action $(\phi^{T^*})^{T^*}$ of G on $T^*(T^*Q)$.

Proof. (i) The Atiyah algebroid associated with the principal bundle $\pi_{T^*} : T^*Q \to T^*Q/G$ is the quotient vector bundle $\tau_{T^*Q}|G : T(T^*Q)/G \to T^*Q/G$ of $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ by the action $(\phi^{T^*})^T$ of G on $T(T^*Q)$.

On the other hand, we have that the fibre of $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ over $[\alpha_q] \in T^*Q/G$, with $\alpha_q \in T_q^*Q$, is the subspace of $(TQ/G)_{\pi(q)} \times T_{[\alpha_q]}(T^*Q/G)$ defined by

$$\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)_{[\alpha_q]} = \big\{\big([v_q], X_{[\alpha_q]}\big) \in (TQ/G)_{\pi(q)} \times T_{[\alpha_q]}(T^*Q/G)/ \\ (T_q\pi)(v_q) = \big(T_{[\alpha_q]}(\pi_Q|G)\big)\big(X_{[\alpha_q]}\big)\big\}.$$

Now, we define the morphism $(\pi_T \circ T\pi_Q, T\pi_{T^*})$ between the vector bundles $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$ over the map $\pi_{T^*} : T^*Q \to T^*Q/G$ as follows,

$$(\pi_T \circ T\pi_Q, T\pi_{T^*})\big(X_{\alpha_q}\big) = \big(\pi_T\big(\big(T_{\alpha_q}\pi_Q\big)\big(X_{\alpha_q}\big)\big), \big(T_{\alpha_q}\pi_{T^*}\big)\big(X_{\alpha_q}\big)\big) \tag{9.7}$$

for $X_{\alpha_q} \in T_{\alpha_q}(T^*Q)$, with $\alpha_q \in T_q^*Q$.

Since the following diagram

$$\begin{tikzcd} T^*Q \arrow[r, "\pi_Q"] \arrow[d, "\pi_{T^*}"'] & Q \arrow[d, "\pi"] \\ T^*Q/G \arrow[r, "\pi_Q|G"] & M = Q/G \end{tikzcd}$$

is commutative, one deduces that

$$(\pi_T \circ T\pi_Q, T\pi_{T^*})\big(X_{\alpha_q}\big) \in \mathcal{L}^{(\tau_Q|G)^*}(TQ/G)_{[\alpha_q]}$$

and, thus, the map $(\pi_T \circ T\pi_Q, T\pi_{T^*})$ is well-defined.

If $\alpha_q \in T_q^*Q$, we will denote by $(\phi^{T^*})_{\alpha_q} : G \to T^*Q$ and by $\phi_q : G \to Q$ the maps given by

$$(\Phi^{T^*})_{\alpha_q}(g) = (\Phi^{T^*})_g(\alpha_q) = (T^*\phi_{g^{-1}})(\alpha_q),$$
$$\phi_q(g) = \phi_g(q) = \phi(g, q),$$

for $g \in G$. Then, proceeding as in the first step of the proof of theorem 9.1 and using that $\pi_Q \circ (\phi^{T^*})_{\alpha_q} = \phi_q$ and the fact that the following diagram

$$\begin{tikzcd} & T^*Q \arrow[dl, "\pi_{T^*}"'] \arrow[r, "\pi_Q"] \arrow[d, "\phi_g^{T^*} = T^*\phi_{g^{-1}}"] & Q \arrow[d, "\phi_g"] \\ T^*Q/G & T^*Q \arrow[l, "\pi_{T^*}"] \arrow[r, "\pi_Q"] & Q \end{tikzcd}$$

is commutative, we deduce that $(\pi_T \circ T\pi_Q, T\pi_{T^*})$ induces an isomorphism $(\pi_T \circ \widetilde{T\pi_Q, T}\pi_{T^*})$ between the vector bundles $\tau_{T^*Q}|G : T(T^*Q)/G \to T^*Q/G$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$.

On the other hand, proceeding as in the second step of the proof of theorem 9.1 and using (2.20), (3.1) and (3.2), we conclude that $(\pi_T \circ \widetilde{T\pi_Q, T}\pi_{T^*})$ is a Lie algebroid isomorphism.

(ii) It follows using (i) and the results of section 2.1.3 (see example 2.3, (*b*)). □

9.2. Lagrangian submanifolds in prolongations of Atiyah algebroids and Hamilton–Poincaré equations

Let $\pi : Q \to M$ be a principal bundle with structural group G, $\phi : G \times Q \to Q$ be the free action of G on Q and $\tau_Q|G : TQ/G \to M$ be the Atiyah algebroid associated with the principal bundle $\pi : Q \to M$. Then, the dual bundle to $\tau_Q|G : TQ/G \to M$ may be identified with the quotient vector bundle $\pi_Q|G : T^*Q/G \to M$ of the cotangent bundle $\pi_Q : T^*Q \to Q$ by the cotangent action ϕ^{T^*} of G on T^*Q.

Now, denote by $(\pi_T \circ T\pi_Q, T\pi_{T^*}) : T(T^*Q) \to \mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ the map given by (9.7). Then, the pair $((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})$ is a morphism between the vector bundles $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$. We remark that $\mathcal{L}^{\pi_Q}(TQ) \cong T(T^*Q)$ and, thus, the Lie algebroids $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$ are symplectic (see section 3.2).

Theorem 9.4. (*i*) *The pair $((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})$ is a symplectomorphism between the symplectic Lie algebroids $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$. In other words, we have:*

(i_a) *The pair $((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})$ is a morphism between the Lie algebroids $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$.*

(i_b) *If Ω_{TQ} (respectively, $\Omega_{TQ/G}$) is the canonical symplectic section of $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ (respectively, $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$) then*

$$((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})^*(\Omega_{TQ/G}) = \Omega_{TQ}. \tag{9.8}$$

(ii) Let $h : T^*Q/G \to \mathbb{R}$ be a Hamiltonian function and $H : T^*Q \to \mathbb{R}$ be the corresponding G-invariant Hamiltonian on T^*Q

$$H = h \circ \pi_{T^*}. \tag{9.9}$$

If $\xi_H \in \Gamma(T(T^*Q)) \cong \mathfrak{X}(T^*Q)$ (respectively, $\xi_h \in \Gamma(\mathcal{L}^{(\tau_Q|G)^*}(TQ/G))$) is the Hamiltonian section associated with H (respectively, h) then

$$(\pi_T \circ T\pi_Q, T\pi_{T^*}) \circ \xi_H = \xi_h \circ \pi_{T^*}.$$

Proof. (i) We consider the Atiyah algebroid $\tau_{T^*Q}|G : T(T^*Q)/G \to T^*Q/G$ associated with the principal bundle $\pi_{T^*} : T^*Q \to T^*Q/G$. If $\pi_{TT^*} : T(T^*Q) \to T(T^*Q)/G$ is the canonical projection, we have that the pair (π_{TT^*}, π_{T^*}) is a Lie algebroid morphism (see section 2.1.3).

Now, denote by $(\pi_T \circ \widetilde{T\pi_Q, T}\pi_{T^*}) : T(T^*Q)/G \to \mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ the isomorphism between the Lie algebroids $\tau_{T^*Q}|G : T(T^*Q)/G \to T^*Q/G$ and $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$ considered in the proof of theorem 9.3. It follows that

$$(\pi_T \circ T\pi_Q, T\pi_{T^*}) = (\pi_T \circ \widetilde{T\pi_Q, T}\pi_{T^*}) \circ \pi_{TT^*}.$$

This proves (i_a).

Next, we will prove (i_b). If λ_{TQ} (respectively, $\lambda_{TQ/G}$) is the Liouville section of $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ (respectively, $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$) then, using (3.4) and (9.7), we deduce that

$$((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})^*(\lambda_{TQ/G}) = \lambda_{TQ}.$$

Thus, from (i_a) and since $\Omega_{TQ/G} = -d^{\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)}\lambda_{TQ/G}$ and $\Omega_{TQ} = -d^{T(T^*Q)}\lambda_{TQ}$, we conclude that

$$((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})^*(\Omega_{TQ/G}) = \Omega_{TQ}.$$

(ii) Using (i_a) and (9.9), we obtain that

$$((\pi_T \circ T\pi_Q, T\pi_{T^*}), \pi_{T^*})^*\big(d^{\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)}h\big) = d^{T(T^*Q)}H. \tag{9.10}$$

Therefore, from (3.10), (9.8) and (9.10), it follows that

$$\big(i_{(\pi_T\circ T\pi_Q, T\pi_{T^*})(\xi_H(\alpha_q))}\Omega_{TQ/G}(\pi_{T^*}(\alpha_q))\big)\big((\pi_T \circ T\pi_Q, T\pi_{T^*})(X_{\alpha_q})\big)$$
$$= \big(i_{\xi_h(\pi_{T^*}(\alpha_q))}\Omega_{TQ/G}(\pi_{T^*}(\alpha_q))\big)\big((\pi_T \circ T\pi_Q, T\pi_{T^*})(X_{\alpha_q})\big),$$

for $\alpha_q \in T_q^*Q$ and $X_{\alpha_q} \in T_{\alpha_q}(T^*Q)$. This implies that

$$(\pi_T \circ T\pi_Q, T\pi_{T^*})(\xi_H(\alpha_q)) = \xi_h(\pi_{T^*}(\alpha_q)). \qquad \square$$

Now, we prove the following result.

Corollary 9.5. *Let $h : T^*Q/G \to \mathbb{R}$ be a Hamiltonian function and $H : T^*Q \to \mathbb{R}$ be the corresponding G-invariant Hamiltonian on T^*Q. Then, the solutions of the Hamilton equations for h are just the solutions of the Hamilton–Poincaré equations for H.*

Proof. We will give two proofs.

First proof (as a consequence of theorem 9.4). Let ρ be the anchor map of the Atiyah algebroid $\tau_Q|G : TQ/G \to M$ and $\rho^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T(T^*Q/G)$ be the anchor map of the Lie algebroid $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$. Using theorem 9.4, we deduce that the vector field $\xi_H \in \mathfrak{X}(T^*Q)$ is π_{T^*}-projectable on the vector field $\rho^{(\tau_Q|G)^*}(\xi_h) \in \mathfrak{X}(T^*Q/G)$. Thus, the projections, via π_{T^*}, of the integral curves of ξ_H are the integral curves of the vector field $\rho^{(\tau_Q|G)^*}(\xi_h)$. But, the integral curves of ξ_H and $\rho^{(\tau_Q|G)^*}(\xi_h)$ are the solutions of the Hamilton equations for H and h, respectively (see section 3.3). Finally, since the projections (via π_{T^*}) of the solutions of the Hamilton equations for H are the solutions of the Hamilton–Poincaré equations for H (see [5]) the result follows.

Second proof (a direct local proof). Let $A : TQ \to \mathfrak{g}$ be a (principal) connection on the principal bundle $\pi : Q \to M$ and $B : TQ \oplus TQ \to \mathfrak{g}$ be the curvature of A. We choose a local trivialization of $\pi : Q \to M$ to be $U \times G$, where U is an open subset of M such that there are local coordinates (x^i) on U. Suppose that $\{\xi_a\}$ is a basis of $\mathfrak{g}$, that c_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_a\}$ and that A_i^a (respectively, B_{ij}^a) are the components of A (respectively, B) with respect to the local coordinates (x^i) and the basis $\{\xi_a\}$ (see (2.18)).

Denote by $\{e_i, e_a\}$ the local basis of G-invariant vector fields on Q given by (2.17), by $(x^i, \dot{x}^i, \bar{v}^a)$ the corresponding local fibred coordinates on TQ/G and by $(x^i, p_i, \bar{p}_a)$ the (dual) local fibred coordinates on T^*Q/G. Then, using (2.20) and (3.12), we derive the Hamilton equations for h

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial h}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial h}{\partial x^i} + B_{ij}^a \bar{p}_a \frac{\partial h}{\partial p_j} - c_{ab}^c A_i^b \bar{p}_c \frac{\partial h}{\partial \bar{p}_a},$$
$$\frac{\mathrm{d}\bar{p}_a}{\mathrm{d}t} = c_{ab}^c A_i^b \bar{p}_c \frac{\partial h}{\partial p_i} - c_{ab}^c \bar{p}_c \frac{\partial h}{\partial \bar{p}_b},$$

which are just the Hamilton–Poincaré equations associated with the G-invariant Hamiltonian H (see [5]). $\square$

As we know (see section 3.1), the local basis $\{e_i, e_a\}$ of $\Gamma(TQ)$ induces a local basis $\{\tilde{e}_i, \tilde{e}_a, \bar{e}_i, \bar{e}_a\}$ of $\Gamma(\mathcal{L}^{(\tau_Q|G)^*}(TQ/G))$ and we may consider the corresponding local coordinates $(x^i, y_i, y_a; z^i, z^a, v_i, v_a)$ on $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ (see again section 3.1).

Since the Lie algebroids $(\tau_Q|G)^{(\tau_Q|G)^*} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to T^*Q/G$ and $\tau_{T^*Q}|G : T(T^*Q)/G \to T^*Q/G$ are isomorphic and $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \subseteq TQ/G \times T(T^*Q/G)$, we will adopt the following notation for the above coordinates:

$$(x^i, p_i, \bar{p}_a; \dot{x}^i, \bar{v}^a, \dot{p}_i, \dot{\bar{p}}_a).$$

We recall that $(x^i, \dot{x}_i, \bar{v}^a)$ and $(x^i, p_i, \bar{p}_a)$ are the local coordinates on TQ/G and T^*Q/G, respectively (see the second proof of corollary 9.5).

Next, using the coordinates $(x^i, p_i, \bar{p}_a; \dot{x}^i, \bar{v}^a, \dot{p}_i, \dot{\bar{p}}_a)$, we will obtain the local equations defining the Lagrangian submanifold $S_h = \xi_h(T^*Q/G)$ of the symplectic Lie algebroid $(\mathcal{L}^{(\tau_Q|G)^*}(TQ/G), \Omega_{TQ/G})$, $h : T^*Q/G \to \mathbb{R}$ being a Hamiltonian function.

Using (2.20) and (3.11), we deduce that the local expression of ξ_h is

$$\begin{aligned}\xi_h(x^i, p_i, \bar{p}_a) = {}& \frac{\partial h}{\partial p_i}\tilde{e}_i + \frac{\partial h}{\partial \bar{p}_a}\tilde{e}_a - \left(\frac{\partial h}{\partial x^i} - B^a_{ij}\bar{p}_a\frac{\partial h}{\partial p_j} - c^a_{bd}A^b_i\bar{p}_a\frac{\partial h}{\partial \bar{p}_d}\right)\bar{e}_i \\ &+ \left(c^c_{ab}A^b_i\bar{p}_c\frac{\partial h}{\partial p_i} - c^c_{ab}\bar{p}_c\frac{\partial h}{\partial \bar{p}_b}\right)\bar{e}_a.\end{aligned} \tag{9.11}$$

Thus, the local equations defining the submanifold S_h are

$$\begin{aligned}\bar{v}^a &= \frac{\partial h}{\partial \bar{p}_a},\\ \dot{x}^i &= \frac{\partial h}{\partial p_i}, \qquad \dot{p}_i = -\left(\frac{\partial h}{\partial x^i} - B^a_{ij}\bar{p}_a\frac{\partial h}{\partial p_j} - c^a_{bd}A^b_i\bar{p}_a\frac{\partial h}{\partial \bar{p}_d}\right),\\ \dot{\bar{p}}_a &= c^c_{ab}A^b_i\bar{p}_c\frac{\partial h}{\partial p_i} - c^c_{ab}\bar{p}_c\frac{\partial h}{\partial \bar{p}_b},\end{aligned}$$

or, in other words,

$$\bar{v}^a = \frac{\partial h}{\partial \bar{p}_a}, \tag{9.12}$$

$$\begin{aligned}\frac{\mathrm{d}x^i}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\left(\frac{\partial h}{\partial x^i} - B^a_{ij}\bar{p}_a\frac{\partial h}{\partial p_j} - c^a_{bd}A^b_i\bar{p}_a\frac{\partial h}{\partial \bar{p}_d}\right),\\ \frac{\mathrm{d}\bar{p}_a}{\mathrm{d}t} &= c^c_{ab}A^b_i\bar{p}_c\frac{\partial h}{\partial p_i} - c^c_{ab}\bar{p}_c\frac{\partial h}{\partial \bar{p}_b}.\end{aligned} \tag{9.13}$$

Equations (9.12) give the definition of the components of the *locked body angular velocity* (in the terminology of [2]) and equations (9.13) are just the *Hamilton–Poincaré equations* for the G-invariant Hamiltonian $H = h \circ \pi_{T^*}$.

Finally, we will discuss the relation between the solutions of the Hamilton–Jacobi equation for the Hamiltonians h and H.

Suppose that $\alpha \in \Gamma(T^*Q/G)$ is a 1-cocycle of the Atiyah algebroid $\tau_Q|G : TQ/G \to M = Q/G$. Then, since the pair (π_T, π) is a morphism between the Lie algebroids $\tau_Q : TQ \to Q$ and $\tau_Q|G : TQ/G \to M = Q/G$ (see section 2.1.3), we deduce that the section $\tilde{\alpha} \in \Gamma(T^*Q)$ given by

$$\tilde{\alpha} = (\pi_T, \pi)^*\alpha, \tag{9.14}$$

is also a 1-cocycle or, in other words, $\tilde{\alpha}$ is a closed 1-form on Q. It is clear that $\tilde{\alpha}$ is G-invariant. Conversely, if $\tilde{\alpha}$ is a G-invariant closed 1-form on Q then, using that $(\pi_T)_{|T_qQ} : T_qQ \to (TQ/G)_{\pi(q)}$ is a linear isomorphism, for all $q \in Q$, we deduce that there exists a unique 1-cocycle $\alpha \in \Gamma(T^*Q/G)$ of the Atiyah algebroid $\tau_Q|G : TQ/G \to M$ such that (9.14) holds.

Proposition 9.6. *There exists a one-to-one correspondence between the solutions of the Hamilton–Jacobi equation for h and the G-invariant solutions of the Hamilton–Jacobi equation for H.*

Proof. We recall that a 1-cocycle $\alpha \in \Gamma(T^*Q/G)$ (respectively, $\tilde{\alpha} \in \Gamma(T^*Q)$) is a solution of the Hamilton–Jacobi equation for h (respectively, H) if $d^{TQ/G}(h \circ \alpha) = 0$ (respectively, $d^{TQ}(H \circ \tilde{\alpha}) = 0$).

Now, assume that α is a 1-cocycle of the Atiyah algebroid $\tau_Q|G : TQ/G \to M$ and denote by $\tilde{\alpha}$ the cocycle defined by (9.14). We obtain that

$$\pi_{T^*} \circ d^{TQ}(H \circ \tilde{\alpha}) = d^{TQ/G}(h \circ \alpha) \circ \pi.$$

Thus, using that

$$\pi_{T^*|T_q^*Q} : T_q^*Q \to (T^*Q/G)_{\pi(q)}$$

is a linear isomorphism, for all $q \in Q$, we conclude that

$$d^{TQ/G}(h \circ \alpha) = 0 \quad \Leftrightarrow \quad d^{TQ}(H \circ \tilde{\alpha}) = 0$$

which proves the result. □

9.3. Lagrangian submanifolds in prolongations of Atiyah algebroids and Lagrange–Poincaré equations

Let $\pi : Q \to M$ be a principal bundle with structural group G, $\tau_Q|G : TQ/G \to M$ be the Atiyah algebroid associated with the principal bundle $\pi : Q \to M$ and $\pi_T : TQ \to TQ/G$ be the canonical projection.

Theorem 9.7. *The solutions of the Euler–Lagrange equations for a Lagrangian $l : TQ/G \to \mathbb{R}$ are the solutions of the Lagrange–Poincaré equations for the corresponding G-invariant Lagrangian L given by $L = l \circ \pi_T$.*

Proof. Let $A : TQ \to \mathfrak{g}$ be a (principal) connection on the principal bundle $\pi : Q \to M$ and $B : TQ \oplus TQ \to \mathfrak{g}$ be the curvature of A. We choose a local trivialization of $\pi : Q \to M$ to be $U \times G$, where U is an open subset on M such that there are local coordinates (x^i) on U. Suppose that $\{\xi_a\}$ is a basis of $\mathfrak{g}$, that c_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_a\}$ and that A_i^a (respectively, B_{ij}^a) are the components of A (respectively, B) with respect to the local coordinates (x^i) and the basis $\{\xi_a\}$ (see (2.18)).

Denote by $\{e_i, e_a\}$ the local basis of G-invariant vector fields on Q given by (2.17) and by $(x^i, \dot{x}^i, \bar{v}^a)$ the corresponding local fibred coordinates on TQ/G. Then, using (2.20) and (2.40), we derive the Euler–Lagrange equations for l

$$\begin{aligned} \frac{\partial l}{\partial x^j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \dot{x}^j}\right) &= \frac{\partial l}{\partial \bar{v}^a}\left(B_{ij}^a \dot{x}^i + c_{db}^a A_j^b \bar{v}^d\right), &&\text{for all} \quad j, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \bar{v}^b}\right) &= \frac{\partial l}{\partial \bar{v}^a}\left(c_{db}^a \bar{v}^d - c_{db}^a A_i^d \dot{x}^i\right), &&\text{for all} \quad b, \end{aligned}$$

which are just the *Lagrange–Poincaré equations* associated with the G-invariant Lagrangian L (see [6]). □

Now, let $A_{TQ/G} : \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \equiv \rho^*(T(T^*Q/G)) \to \mathcal{L}^{(\tau_Q|G)}(TQ/G)^*$ be the isomorphism between the vector bundles $pr_1 : \rho^*(T(T^*Q/G)) \equiv \mathcal{L}^{(\tau_Q|G)^*}(TQ/G) \to TQ/G$ and $((\tau_Q|G)^{(\tau_Q|G)})^* : \mathcal{L}^{(\tau_Q|G)}(TQ/G)^* \to TQ/G$ defined in section 5 (see (5.5)) and $\Omega_{TQ/G}$ be the canonical symplectic section associated with the Atiyah algebroid $\tau_Q|G : TQ/G \to M$.

Next, we will obtain the local equations defining the Lagrangian submanifold $S_l = \big(A^{-1}_{TQ/G} \circ d^{\mathcal{L}^{(\tau_Q|G)}(TQ/G)} l\big)(TQ/G)$ of the symplectic Lie algebroid $(\mathcal{L}^{(\tau_Q|G)^*}(TQ/G), \Omega_{TQ/G})$.

The local basis $\{e_i, e_a\}$ induces a local basis $\{\tilde{T}_i, \tilde{T}_a, \tilde{V}_i, \tilde{V}_a\}$ of $\Gamma(\mathcal{L}^{(\tau_Q|G)}(TQ/G))$ (see remark 2.7) and we may consider the corresponding local coordinates $(x^i, \dot{x}^i, \bar{v}^a; z^i, z^a, v^i, v^a)$ on $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ (see again remark 2.7). We will denote by $(x^i, \dot{x}^i, \bar{v}^a; z_i, z_a, v_i, v_a)$ the dual coordinates on the dual bundle $\mathcal{L}^{(\tau_Q|G)}(TQ/G)^*$ to $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$.

On the other hand, we will use the notation of section 9.2 for the local coordinates on $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$, that is, $(x^i, p_i, \bar{p}_a; \dot{x}^i, \bar{v}^a, \dot{p}_i, \dot{\bar{p}}_a)$.

Then, from (2.20), (2.36) and (5.5), we deduce that

$$\big(d^{\mathcal{L}^{(\tau_Q|G)}(TQ/G)} l\big)(x^i, \dot{x}^i, \bar{v}^a) = \frac{\partial l}{\partial x^i}\tilde{T}^i + \frac{\partial l}{\partial \dot{x}^i}\tilde{V}^i + \frac{\partial l}{\partial \bar{v}^a}\tilde{V}^a,$$

$$A^{-1}_{TQ/G}(x^i, \dot{x}^i, \bar{v}^a; z_i, z_a, v_i, v_a) = \big(x^i, v_i, v_a; \dot{x}^i, \bar{v}^a, z_i + B^c_{ij}\dot{x}^j v_c - c^c_{ab}A^b_i v_c \bar{v}^a,$$
$$z_a - c^c_{ab}\bar{v}^b v_c + c^c_{ab}A^b_j \dot{x}^j v_c\big),$$

where $\{\tilde{T}^i, \tilde{T}^a, \tilde{V}^i, \tilde{V}^a\}$ is the dual basis of $\{\tilde{T}_i, \tilde{T}_a, \tilde{V}_i, \tilde{V}_a\}$.

Thus, the local equations defining the Lagrangian submanifold S_l of the symplectic Lie algebroid $(\mathcal{L}^{(\tau_Q|G)^*}(TQ/G), \Omega_{TQ/G})$ are

$$p_i = \frac{\partial l}{\partial \dot{x}^i}, \qquad \bar{p}_a = \frac{\partial l}{\partial \bar{v}^a},$$
$$\dot{p}_i = \frac{\partial l}{\partial x^i} + B^a_{ij}\dot{x}^j \frac{\partial l}{\partial \bar{v}^a} - c^a_{db}A^b_i \bar{v}^d \frac{\partial l}{\partial \bar{v}^a},$$
$$\dot{\bar{p}}_b = \frac{\partial l}{\partial \bar{v}^a}\left(c^a_{db}\bar{v}^d - c^a_{db}A^d_i \dot{x}^i\right),$$

or, in other words,

$$p_i = \frac{\partial l}{\partial \dot{x}^i}, \qquad \bar{p}_a = \frac{\partial l}{\partial \bar{v}^a}, \tag{9.15}$$

$$\begin{aligned} \frac{\partial l}{\partial x^j} - \frac{\mathrm{d}p_j}{\mathrm{d}t} &= \bar{p}_a\left(B^a_{ij}\dot{x}^i + c^a_{db}A^b_j \bar{v}^d\right), \\ \frac{\mathrm{d}p_b}{\mathrm{d}t} &= \bar{p}_a\left(c^a_{db}\bar{v}^d - A^d_i c^a_{db}\dot{x}^i\right). \end{aligned} \tag{9.16}$$

Equations (9.15) give the definition of the momenta and equations (9.16) are just the Lagrange–Poincaré equations for the G-invariant Lagrangian L.

Now, let $(\pi_T \circ T\tau_Q, T\pi_T) : T(TQ) \to \mathcal{L}^{(\tau_Q|G)}(TQ/G)$ be the map given by (9.1). Then, the pair $((\pi_T \circ T\tau_Q, T\pi_T), \pi_T)$ is a morphism between the vector bundles $\tau_{TQ} : T(TQ) \to TQ$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$. We recall that the Lie algebroids $\tau_{TQ} : T(TQ) \to TQ$ and $\tau_Q^{\tau_Q} : \mathcal{L}^{\tau_Q}(TQ) \to TQ$ are isomorphic.

Theorem 9.8. *(i) The pair $((\pi_T \circ T\tau_Q, T\pi_T), \pi_T)$ is a morphism between the Lie algebroids $\tau_{TQ} \cong \tau_Q^{\tau_Q} : T(TQ) \cong \mathcal{L}^{\tau_Q}(TQ) \to TQ$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$.*

(ii) Let $l : TQ/G \to \mathbb{R}$ be a Lagrangian function and $L : TQ \to \mathbb{R}$ be the corresponding G-invariant Lagrangian on TQ, $L = l \circ \pi_T$. If $\omega_l \in \Gamma(\wedge^2(\mathcal{L}^{(\tau_Q|G)}(TQ/G)^*))$ and $E_l \in C^\infty(TQ/G)$ (respectively, $\omega_L \in \Gamma(\wedge^2(\mathcal{L}^{\tau_Q}(TQ)^*)) \cong \Gamma(\wedge^2(T^*(TQ)))$ and $E_L \in C^\infty(TQ)$) are the Poincaré–Cartan 2-section and the Lagrangian energy associated with l (respectively, L) then

$$((\pi_T \circ T\tau_Q, T\pi_T), \pi_T)^*(\omega_l) = \omega_L, \tag{9.17}$$

$$E_l \circ \pi_T = E_L. \tag{9.18}$$

(iii) If $Leg_l : TQ/G \to T^*Q/G$ *(respectively,* $Leg_L : TQ \to T^*Q$*) is the Legendre transformation associated with* l *(respectively,* L*) then*

$$Leg_l \circ \pi_T = \pi_{T^*} \circ Leg_L,$$

that is, the following diagram is commutative

$$\begin{tikzcd} TQ \arrow[r, "Leg_L"] \arrow[d, "\pi_T"'] & T^*Q \arrow[d, "\pi_{T^*}"] \\ TQ/G \arrow[r, "Leg_l"] & T^*Q/G \end{tikzcd}$$

Proof. (i) We consider the Atiyah algebroid $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ associated with the principal bundle $\pi_T : TQ \to TQ/G$. If $\pi_{TT} : T(TQ) \to T(TQ)/G$ is the canonical projection, we have that the pair (π_{TT}, π_T) is a Lie algebroid morphism (see section 2.1.3).

Now, denote by $\widetilde{(\pi_T \circ T\tau_Q, T\pi_T)} : T(TQ)/G \to \mathcal{L}^{(\tau_Q|G)}(TQ/G)$ the isomorphism between the Lie algebroids $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$ considered in the proof of theorem 9.1. It follows that

$$(\pi_T \circ T\tau_Q, T\pi_T) = \widetilde{(\pi_T \circ T\tau_Q, T\pi_T)} \circ \pi_{TT}.$$

This proves (i).

(ii) From (2.35), (9.6) and remark 9.2, we deduce that the following diagram is commutative

$$\begin{tikzcd} T(TQ) \arrow[r, "S^{TQ}"] \arrow[d, "(\pi_T \circ T\tau_Q, T\pi_T)"'] & T(TQ) \arrow[d, "(\pi_T \circ T\tau_Q, T\pi_T)"] \\ \mathcal{L}^{(\tau_Q|G)}(TQ/G) \arrow[r, "S^{TQ/G}"] & \mathcal{L}^{(\tau_Q|G)}(TQ/G) \end{tikzcd}$$

where S^{TQ} (respectively, $S^{TQ/G}$) is the vertical endomorphism associated with the Lie algebroid $\tau_Q : TQ \to Q$ (respectively, the Atiyah algebroid $\tau_Q|G : TQ/G \to M$). Thus, if θ_L (respectively, θ_l) is the Poincaré–Cartan 1-section associated with L (respectively, l) then, using the first part of the theorem, (2.37) and the fact that $L = l \circ \pi_T$, we obtain that

$$((\pi_T \circ T\tau_Q, T\pi_T), \pi_T)^*(\theta_l) = \theta_L. \tag{9.19}$$

Therefore, using again the first part of the theorem and (2.38), it follows that

$$((\pi_T \circ T\tau_Q, T\pi_T), \pi_T)^*(\omega_l) = \omega_L.$$

On the other hand, from (2.29), (9.6) and remark 9.2, we have the following diagram is commutative

$$\begin{tikzcd} TQ \arrow[r, "\Delta^{TQ}"] \arrow[d, "\pi_T"'] & T(TQ) \arrow[d, "(\pi_T \circ T\tau_Q, T\pi_T)"] \\ TQ/G \arrow[r, "\Delta^{TQ/G}"] & \mathcal{L}^{(\tau_Q|G)}(TQ/G) \end{tikzcd}$$

where Δ^{TQ} (respectively, $\Delta^{TQ/G}$) is the Euler section associated with the Lie algebroid $\tau_Q : TQ \to Q$ (respectively, the Atiyah algebroid $\tau_Q|G : TQ/G \to M$). Consequently, using the first part of the theorem and the fact that $L = l \circ \pi_T$, we conclude that

$$E_l \circ \pi_T = E_L.$$

(iii) From (3.23) and (9.19), we deduce the result. □

Now we prove the following

Corollary 9.9. *Let $l : TQ/G \to \mathbb{R}$ be a Lagrangian function and $L : TQ \to \mathbb{R}$ be the corresponding G-invariant Lagrangian on TQ, $L = l \circ \pi_T$. Then, L is regular if and only if l is regular.*

Proof. The map

$$(\pi_T \circ T\tau_Q, T\pi_T)_{|T_{v_q}(TQ)} : T_{v_q}(TQ) \to \mathcal{L}^{(\tau_Q|G)}(TQ/G)_{[v_q]}$$

is a linear isomorphism, for all $v_q \in T_q Q$ (see the proof of theorem 9.1).

On the other hand, L (respectively, l) is regular if and only if ω_L (respectively, ω_l) is a symplectic section of $\tau_{TQ} : T(TQ) \to TQ$ (respectively, $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$).

Thus, using (9.17), the result follows. □

Assume that the Lagrangian function $l : TQ/G \to \mathbb{R}$ is regular and denote by $\xi_l \in \Gamma(\mathcal{L}^{(\tau_Q|G)}(TQ/G))$ the Euler–Lagrange section associated with l. We recall that ξ_l is characterized by the equation

$$i_{\xi_l}\omega_l = d^{\mathcal{L}^{(\tau_Q|G)}(TQ/G)} E_l.$$

Next, we will obtain the local equations defining the Lagrangian submanifold $S_{\xi_l} = \xi_l(TQ/G)$ of the symplectic Lie algebroid $\left(\mathcal{L}^{(\tau_Q|G)}(TQ/G), \omega_l^{\mathbf{c}}\right)$, $\omega_l^{\mathbf{c}}$ being the complete lift of ω_l.

Let $A : TQ \to \mathfrak{g}$ be a (principal) connection on the principal bundle $\pi : Q \to M$ and $B : TQ \oplus TQ \to \mathfrak{g}$ be the curvature of A. We choose a local trivialization of $\pi : Q \to M$ to be $U \times G$, where U is an open subset of M such that there are local coordinates (x^i) on U. Suppose that $\{\xi_a\}$ is a basis of $\mathfrak{g}$, that c_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_a\}$ and that A_i^a (respectively, B_{ij}^a) are the components of A (respectively, B) with respect to the local coordinates (x^i) and the basis $\{\xi_a\}$ (see (2.18)).

Denote by $\{e_i, e_a\}$ the local basis of G-invariant vector fields on Q given by (2.17) and by $(x^i, \dot{x}^i, \bar{v}^a)$ the corresponding local fibred coordinates on TQ/G. $\{e_i, e_a\}$ induces a local basis $\{\tilde{T}_i, \tilde{T}_a, \tilde{V}_i, \tilde{V}_a\}$ of $\Gamma(\mathcal{L}^{(\tau_Q|G)}(TQ/G))$ (see (9.4)) and we have the corresponding local coordinates $(x^i, \dot{x}^i, \bar{v}^a; z^i, z^a, v^i, v^a)$ on $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$. Since the vector bundles $\tau_{TQ}|G : T(TQ)/G \to TQ/G$ and $(\tau_Q|G)^{(\tau_Q|G)} : \mathcal{L}^{(\tau_Q|G)}(TQ/G) \to TQ/G$ are isomorphic (see theorem 9.1), we will adopt the following notation for the above coordinates

$$(x^i, \dot{x}^i, \bar{v}^a; z^i, z^a, \ddot{x}^i, \dot{\bar{v}}^a).$$

Now, we consider the regular matrix

$$\begin{pmatrix} W_{ij} & W_{ia} \\ W_{ai} & W_{ab} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 l}{\partial \dot{x}^i \partial \dot{x}^j} & \dfrac{\partial^2 l}{\partial \dot{x}^i \partial \bar{v}^a} \\ \dfrac{\partial^2 l}{\partial \bar{v}^a \partial \dot{x}^i} & \dfrac{\partial^2 l}{\partial \bar{v}^a \partial \bar{v}^b} \end{pmatrix}$$

and denote by

$$\begin{pmatrix} W^{ij} & W^{ia} \\ W^{ai} & W^{ab} \end{pmatrix}$$

the inverse matrix. Then, from (2.20) and (2.42), we deduce that

$$\xi_l = \dot{x}^i \tilde{T}_i + \bar{v}^a \tilde{T}_a + (W^{ij}(\xi_l)_j + W^{ia}(\xi_l)_a)\tilde{V}_i + (W^{ai}(\xi_l)_i + W^{ab}(\xi_l)_b)\tilde{V}_a, \tag{9.20}$$

where

$$\begin{aligned} (\xi_l)_i &= \frac{\partial l}{\partial x^i} - \frac{\partial^2 l}{\partial x^j \partial \dot{x}^i}\dot{x}^j + \frac{\partial l}{\partial \bar{v}^a}\left(B^a_{ij}\dot{x}^j + c^a_{bd}\bar{v}^b A^d_i\right), \\ (\xi_l)_b &= -\frac{\partial^2 l}{\partial x^i \partial \bar{v}^b}\dot{x}^i + \frac{\partial l}{\partial \bar{v}^c}\left(c^c_{bd} A^d_i \dot{x}^i + c^c_{ab}\bar{v}^a\right). \end{aligned} \tag{9.21}$$

Thus, using the coordinates $(x^i, \dot{x}^i, \bar{v}^a; z^i, z^a, \ddot{x}^i, \dot{\bar{v}}^a)$, we obtain that the local equations defining the Lagrangian submanifold S_{ξ_l} are

$$z^i = \dot{x}^i, \qquad z^a = \bar{v}^a, \tag{9.22}$$

$$\ddot{x}^i = W^{ij}(\xi_l)_j + W^{ib}(\xi_l)_b, \qquad \dot{\bar{v}}^a = W^{aj}(\xi_l)_j + W^{ab}(\xi_l)_b. \tag{9.23}$$

From (9.21) and (9.23), we conclude that

$$\begin{aligned} \frac{\partial l}{\partial x^j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \dot{x}^j}\right) &= \frac{\partial l}{\partial \bar{v}^a}\left(B^a_{ij}\dot{x}^i + c^a_{db}\bar{v}^d A^b_j\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \bar{v}^b}\right) &= \frac{\partial l}{\partial \bar{v}^a}\left(c^a_{db}\bar{v}^d - c^a_{db}A^d_i\dot{x}^i\right), \end{aligned}$$

which are just the Lagrange–Poincaré equations associated with the G-invariant Lagrangian $L = l \circ \pi_T$.

9.4. A particular example: Wong's equations

To illustrate the theory that we have developed in this section, we will consider an interesting example, that of *Wong's equations*. Wong's equations arise in at least two different interesting contexts. The first of these concerns the dynamics of a coloured particle in a Yang–Mills field and the second one is that of the falling cat theorem (see [37–39]; see also [6] and references quoted therein).

Let (M, g_M) be a given Riemannian manifold, G be a compact Lie group with a bi-invariant Riemannian metric κ and $\pi : Q \to M$ be a principal bundle with structure group G. Suppose that $\mathfrak{g}$ is the Lie algebra of G, that $A : TQ \to \mathfrak{g}$ is a principal connection on Q and that $B : TQ \oplus TQ \to \mathfrak{g}$ is the curvature of A.

If $q \in Q$ then, using the connection A, one may prove that the tangent space to Q at q, T_qQ, is isomorphic to the vector space $\mathfrak{g} \oplus T_{\pi(q)}M$. Thus, κ and g_M induce a Riemannian metric g_Q on Q and we can consider the kinetic energy $L : TQ \to \mathbb{R}$ associated with g_Q. The Lagrangian L is given by

$$L(v_q) = \tfrac{1}{2}(\kappa_e(A(v_q), A(v_q)) + g_{\pi(q)}((T_q\pi)(v_q), (T_q\pi)(v_q))),$$

for $v_q \in T_qQ$, e being the identity element in G. It is clear that L is hyperregular and G-invariant.

On the other hand, since the Riemannian metric g_Q is also G-invariant, it induces a fibre metric $g_{TQ/G}$ on the quotient vector bundle $\tau_Q|G : TQ/G \to M = Q/G$. The reduced Lagrangian $l : TQ/G \to \mathbb{R}$ is just the kinetic energy of the fibre metric $g_{TQ/G}$, that is,

$$l[v_q] = \tfrac{1}{2}(\kappa_e(A(v_q), A(v_q)) + g_{\pi(q)}((T_q\pi)(v_q), (T_q\pi)(v_q))),$$

for $v_q \in T_qQ$.

We have that l is hyperregular. In fact, the Legendre transformation associated with l is just the vector bundle isomorphism $\flat_{g_{TQ/G}}$ between TQ/G and T^*Q/G induced by the fibre metric $g_{TQ/G}$. Thus, the reduced Hamiltonian $h : T^*Q/G \to \mathbb{R}$ is given by

$$h[\alpha_q] = l\big(\flat^{-1}_{g_{TQ/G}}[\alpha_q]\big)$$

for $\alpha_q \in T_q^*Q$.

Now, we choose a local trivialization of $\pi : Q \to M$ to be $U \times G$, where U is an open subset of M such that there are local coordinates (x^i) on U. Suppose that $\{\xi_a\}$ is a basis of $\mathfrak{g}$, that c_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_a\}$, that A_i^a (respectively, B_{ij}^a) are the components of A (respectively, B) with respect to the local coordinates (x^i) and the basis $\{\xi_a\}$ (see (2.18)) and that

$$\kappa_e = \kappa_{ab}\xi^a \otimes \xi^b, \qquad\qquad g = g_{ij}dx^i \otimes dx^j,$$

where $\{\xi^a\}$ is the dual basis to $\{\xi_a\}$. Note that since κ is a bi-invariant metric on G, it follows that

$$c_{ab}^c\kappa_{cd} = c_{ad}^c\kappa_{cb}. \tag{9.24}$$

Denote by $\{e_i, e_a\}$ the local basis of G-invariant vector fields on Q given by (2.17), by $(x^i, \dot{x}^i, \bar{v}^a)$ the corresponding local fibred coordinates on TQ/G and by $(x^i, p_i, \bar{p}_a)$ the (dual) coordinates on T^*Q/G. We have that

$$l(x^i, \dot{x}^i, \bar{v}^a) = \tfrac{1}{2}(\kappa_{ab}\bar{v}^a\bar{v}^b + g_{ij}\dot{x}^i\dot{x}^j), \tag{9.25}$$

$$h(x^i, p_i, \bar{p}_a) = \tfrac{1}{2}(\kappa^{ab}\bar{p}_a\bar{p}_b + g^{ij}p_ip_j), \tag{9.26}$$

where (κ^{ab}) (respectively, (g^{ij})) is the inverse matrix of (κ_{ab}) (respectively, (g_{ij})). Thus, the Hessian matrix of l, W_l, is

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & \kappa_{ab} \end{pmatrix}$$

and the inverse matrix of W_l is

$$\begin{pmatrix} g^{ij} & 0 \\ 0 & \kappa^{ab} \end{pmatrix}$$

The local basis $\{e_i, e_a\}$ induces a local basis $\{\tilde{e}_i, \tilde{e}_a, \bar{e}_i, \bar{e}_a\}$ of $\Gamma(\mathcal{L}^{(\tau_Q|G)^*}(TQ/G))$ and we may consider the corresponding local coordinates

$$(x^i, p_i, \bar{p}_a; \dot{x}^i, \bar{v}^a, \dot{p}_i, \dot{\bar{p}}_a)$$

on $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ (see section 9.2).

From (9.24) and (9.26), we deduce that

$$c_{ab}^c\bar{p}_c\frac{\partial h}{\partial \bar{p}_b} = c_{ab}^c\kappa^{db}\bar{p}_c\bar{p}_d = 0.$$

Thus, the local expression of the Hamiltonian section ξ_h of $\mathcal{L}^{(\tau_Q|G)^*}(TQ/G)$ is (see (9.11) and (9.26))

$$\begin{aligned}\xi_h(x^i, p_i, \bar{p}_a) = (g^{ij}p_j)\tilde{e}_i + (\kappa^{bc}\bar{p}_c)\tilde{e}_b - \left(\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i}p_jp_k - B_{ij}^a\bar{p}_ag^{jk}p_k\right)\bar{e}_i \\ - \left(c_{ab}^cA_i^a\bar{p}_cg^{ij}p_j\right)\bar{e}_b.\end{aligned}$$

Therefore, the local equations defining the Lagrangian submanifold $S_l = S_h = \xi_h(T^*Q/G)$ are

$$\dot{x}^i = g^{ij} p_j, \qquad \bar{v}^b = \kappa^{bc}\bar{p}_c, \qquad \text{for all} \quad i \text{ and } b,$$
$$\dot{p}_i = -\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i} p_j p_k - \bar{p}_a B^a_{ji} g^{jk} p_k, \qquad \text{for all} \quad i,$$
$$\dot{\bar{p}}_b = -c^a_{db} A^d_i \bar{p}_a g^{ij} p_j, \qquad \text{for all} \quad b$$

or, in other words,

$$\dot{x}^i = g^{ij} p_j, \qquad \bar{v}^b = \kappa^{bc}\bar{p}_c, \qquad \text{for all} \quad i \text{ and } b,$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i} p_j p_k - \bar{p}_a B^a_{ji} g^{jk} p_k, \qquad \text{for all} \quad i, \tag{9.27}$$
$$\frac{\mathrm{d}\bar{p}_b}{\mathrm{d}t} = -c^a_{db} A^d_i \bar{p}_a \dot{x}^i, \qquad \text{for all} \quad b. \tag{9.28}$$

Equations (9.27) (respectively, equations (9.28)) are the *second* (respectively, *first*) *Wong equation* (see [6]).

On the other hand, the local basis $\{e_i, e_a\}$ induces a local basis $\{\tilde{T}_i, \tilde{T}_a, \tilde{V}_i, \tilde{V}_a\}$ of $\Gamma(\mathcal{L}^{(\tau_Q|G)}(TQ/G))$ and we may consider the corresponding local coordinates

$$(x^i, \dot{x}^i, \bar{v}^a; z^i, z^a, \ddot{x}^i, \dot{\bar{v}}^a)$$

on $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ (see section 9.3).

From (9.20), (9.21), (9.24) and (9.25), we obtain that the Euler–Lagrange section ξ_l associated with l is given by

$$\xi_l(x^k, \dot{x}^k, \bar{v}^c) = \dot{x}^i \tilde{T}_i + \bar{v}^b \tilde{T}_b + g^{ij}\left(\frac{1}{2}\frac{\partial g_{kl}}{\partial x^j}\dot{x}^k\dot{x}^l - \frac{\partial g_{jk}}{\partial x^l}\dot{x}^k\dot{x}^l + \kappa_{ab}\bar{v}^b B^a_{jk}\dot{x}^k\right)\tilde{V}_i$$
$$+ \left(c^b_{ac}\bar{v}^a A^c_i \dot{x}^i\right)\tilde{V}_b.$$

Thus, the local equations defining the Lagrangian submanifold $S_{\xi_l} = \xi_l(TQ/G)$ of $\mathcal{L}^{(\tau_Q|G)}(TQ/G)$ are

$$z^i = \dot{x}^i, \qquad z^b = \bar{v}^b, \qquad \text{for all} \quad i \text{ and } b,$$
$$\ddot{x}^i = g^{ij}\left(\frac{1}{2}\frac{\partial g_{kl}}{\partial x^j}\dot{x}^k\dot{x}^l - \frac{\partial g_{jk}}{\partial x^l}\dot{x}^k\dot{x}^l + \kappa_{ab}\bar{v}^b B^a_{jk}\dot{x}^k\right), \qquad \text{for all} \quad i, \tag{9.29}$$
$$\dot{\bar{v}}^b = c^b_{ac}\bar{v}^a A^c_i \dot{x}^i, \qquad \text{for all} \quad b. \tag{9.30}$$

Now, put

$$\bar{p}_b = \kappa_{bc}\bar{v}^c, \qquad p_i = g_{ij}\dot{x}^j,$$

for all b and i.

Then, from (9.24) and (9.30), it follows that

$$\frac{\mathrm{d}\bar{p}_b}{\mathrm{d}t} = -\bar{p}_a c^a_{db} A^d_i \dot{x}^i, \qquad \text{for all} \quad b,$$

which is the first Wong equation. In addition, since (g^{ij}) is the inverse matrix of (g_{ij}), we deduce that

$$\frac{\partial g_{kl}}{\partial x^j}\dot{x}^k\dot{x}^l = -\frac{\partial g^{kl}}{\partial x^j} p_k p_l.$$

Therefore, using (9.24) and (9.29), we conclude that

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i} p_j p_k - \bar{p}_a B^a_{ji} g^{jk} p_k, \qquad \text{for all} \quad i,$$

which is the second Wong equation.

10. Future work

In this section, we propose several problems related to the topics discussed in this paper, which we are working on nowadays or which study will be carried out in the near future:

- *Mechanics on Lie algebroids*. In section 3.7, we introduced the Hamilton–Jacobi equation for a Hamiltonian system on a Lie algebroid and we proved that knowing one solution of the Hamilton–Jacobi equation simplifies the search of trajectories for the corresponding Hamiltonian vector field. So, it would be interesting to continue our inquiries about the Hamilton–Jacobi theory for Hamiltonian systems on Lie algebroids. In particular, it would be interesting to introduce a suitable definition of a local (global) complete integral of the Hamilton–Jacobi equation in such a way that knowing an integral of the equation permits a 'direct determination' of some integral curves of the corresponding Hamiltonian vector field.

  Other goal we have proposed is to develop a geometric formalism for nonholonomic mechanics on Lie algebroids (see [7]). In addition, it would be interesting to develop a geometric formalism for vakonomic mechanics and its applications to control theory. In this direction, in [8] the authors study the accessibility and controllability of mechanical control systems on Lie algebroids. This class of systems includes mechanical systems subject to nonholonomic constraints. Moreover, in [31] a geometric setting for the Pontryagin maximum principle in optimal control theory and in the framework of Lie algebroids is provided. This gives a way to study reduction by symmetry groups of the maximum principle. More recently, in [36] the authors consider Lagrangian systems on Lie algebroids with linear nonholonomic constraints.
- *Mechanics on Lie affgebroids*. In [35] (see also [15, 16, 32, 41]) the authors proposed a possible generalization of the notion of a Lie algebroid to affine bundles in order to build a geometrical model for a time-dependent version of Lagrange (Hamilton) equations on Lie algebroids. The resultant mathematical structures are called Lie affgebroids in the terminology of [15]. More recently, in [27], it was proved that a Lie affgebroid and a Hamiltonian section on it induce an analogous to Tulczyjew's triple associated with a Lie algebroid. This construction may be applied in order to give some interesting descriptions of Lagrangian (Hamiltonian) mechanics on Lie affgebroids (see [18]). Symplectic Lie affgebroids and Lagrangian submanifolds of them play an important role in this theory. On the other hand, the notion of a Lagrangian submanifold of a symplectic Lie affgebroid could be used in order to develop a Hamilton–Jacobi theory on Lie affgebroids.

  A different aspect we can work on it is nonholonomic and vakonomic mechanics on Lie affgebroids.
- *Discrete mechanics on Lie groupoids*. In his paper, Weinstein [50] introduced the discrete Euler–Lagrange equations for a Lagrangian function $L : G \to \mathbb{R}$ on a Lie groupoid G. The solutions of these equations are the extremals for a variational principle. In addition, these equations may be considered as the discrete version of the Euler–Lagrange equations on Lie algebroids. Note that Lie algebroids are the infinitesimal invariants of Lie groupoids. After introducing the discrete Euler–Lagrange equations, Weinstein poses the question of developing a Lagrangian (Hamiltonian) formalism on general Lie groupoids. Some progress has been made in that direction (see [28]). In fact, if $L : G \to \mathbb{R}$ is a discrete Lagrangian, we have proved that the appropriate spaces in order to develop the above formalisms are the prolongation of order 1 of G (in the sense of Saunders [42]) and the prolongation of the Lie algebroid $\tau : AG \to M$ associated with G over the vector bundle projection $\tau^* : A^*G \to M$ (for more details, see [28]). Probably, the above formalisms will allow us to deal with other problems such as the construction of a discrete

analogue for Lie groupoids of the Tulczyjew's triple associated with a Lie algebroid. This construction could be used in order to give some interesting descriptions of discrete mechanics on Lie groupoids. We also expect to study numerical aspects of the theory. Finally, another interesting goal could be to develop a discrete version of the Lagrangian (Hamiltonian) formalism on Lie affgebroids.

- *Classical field theory and Lie algebroids.* Recently, Martínez [33] (see also [34]) formulated the classical field theory in the setting of Lie algebroids using a multisymplectic approach. He considered an epimorphism of Lie algebroids $\pi : E \to F$ and an affine bundle $\mathcal{J}\pi$ associated with π. $\mathcal{J}\pi$ is the space where the Lagrangian formalism is based. $\mathcal{J}\pi$ is the set of the linear maps from a fibre of F to a fibre of E which are sections of the projection π. In the affine dual of $\mathcal{J}\pi$ there exists a canonical multisymplectic section, which allows us to define a Hamiltonian formalism. This general setting includes several physical theories as particular cases.

  On the other hand, as we have previously said, in [32, 35] the authors have introduced the notion of a Lie affgebroid structure and developed a Lagrangian (Hamiltonian) formalism on Lie affgebroids, which generalizes some classical formalisms for time-dependent mechanics and, in addition, it may be applied to other situations. Since time-dependent mechanics is a one-dimensional field theory, it would be interesting to define the notion of a 'Lie multialgebroid', as a generalization of the notion of a Lie affgebroid, in such a way that this mathematical object encodes the geometric structure necessary to develop field theories. The first example of a Lie multialgebroid should be $\mathcal{J}\pi$. The notion of a Lie multialgebroid will possibly allow us to study other aspects of the theory as Tulczyjew's triples associated with a Lie multialgebroid and Hamilton–Jacobi equation for classical field theories on Lie multialgebroids among others.

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